⑯回 非可換 (行列の) ベリー接続

復習

たとえば $H|\psi\rangle = |\psi\rangle E \Rightarrow A = \langle\psi|d\psi\rangle = A_i dx_i$

一般に

$i\gamma = \int_C A = \oint A_i dx_i$

$C = \frac{1}{2\pi i}\int dA = \frac{1}{2\pi i}\int \langle d\psi|d\psi\rangle$

T^2 // コンパクト

$dA = \langle d\psi|d\psi\rangle + \underbrace{\langle d\psi|d^2\psi\rangle}_{=0}$

$\because\ |d\psi\rangle = dx_i|\partial_i\psi\rangle$

$|d^2\psi\rangle = dx_i\, dx_j\, |\partial_j\partial_i\psi\rangle$

$= \sum_{i<j}\left(dx_i dx_j |\partial_j\partial_i\psi\rangle + dx_j dx_i |\partial_i\partial_j\psi\rangle\right)$

$-dx_i dx_j$

$= 0$

$\langle d\psi|d\psi\rangle$

$= dx_i \langle\partial_i\psi|\, dx_j |\partial_j\psi\rangle$

$= \sum_{i<j} dx_i dx_j \left\{\langle\partial_i\psi|\partial_j\psi\rangle - \langle\partial_j\psi|\partial_i\psi\rangle\right\}$

多成分 $\psi = (|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_M\rangle)$ M コあつめた (縮退してもよい)

$$H|\psi_i\rangle = |\psi_i\rangle E_i \qquad H \to |\psi_i\rangle$$

線型結合の自由度を与える (縮退していれば must)

$$|\psi'_i\rangle = \sum_j |\psi_j\rangle g_{ji} \qquad g_{ji} : \text{係数}$$

$$(|\psi'_1\rangle, \cdots, |\psi'_M\rangle) = (|\psi_1\rangle, \cdots, |\psi_M\rangle) \underbrace{\boxed{\ g\ }}_{M \times M}$$

$$\psi' = \psi g$$

$$\langle\psi_i|\psi_j\rangle = \delta_{ij}$$

$$\langle\psi'_i|\psi'_j\rangle = \delta_{ij} \to g^{\dagger} g = E_M, \quad g^{\dagger} = g^{-1} \quad \text{ユニタリ}$$

c.f. 一成分 $|\psi\rangle \to |\psi'\rangle = |\psi\rangle g$, $g = e^{i\theta}$ 位相除

多成分 $\psi \to \psi' = \psi g$, $g \in U(M)$

1成分 $A = \langle\psi|d\psi\rangle$

多成分 $A = \psi^\dagger d\psi = \begin{pmatrix} \langle\psi_1| \\ \vdots \\ \langle\psi_M| \end{pmatrix} \left(|d\psi_1\rangle, \dots, |d\psi_M\rangle \right)$

$= \underbrace{\boxed{}}_{M \times M}$

$A_{ij} = \langle\psi_i|d\psi_j\rangle$

gauge 変換

$\psi_g = \psi g$ $\quad d\psi_g = d\psi\, g + \psi\, dg$

$A_g = \psi_g^\dagger d\psi_g = \underbrace{g^\dagger \psi^\dagger}_{\psi_g^\dagger} (d\psi\, g + \psi\, dg)$

$= g^\dagger A g + g^\dagger dg = g^{-1} A g + g^{-1} dg$

$A_g = g^{-1} A g + g^{-1} dg$

物理的な必要性

多電子系の状態 $|G\rangle$, $A = \langle G | dG \rangle$, トポロジカル相で必要

free fermion 相互作用なし

$|G\rangle \leftarrow$ 1体問題から再構成

$$H = \sum_{ij} t_{ij} c_i^\dagger c_j = c^\dagger h c$$

$c = \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix}$, N : # atomic sites

$h^\dagger = h$ $N \times N$ 行列

$h \phi_\ell = \phi_\ell \varepsilon_\ell$ ε_ℓ: 1粒子エネルギー

$$\varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_M \le \varepsilon_F \le \varepsilon_{M+1} \cdots \le \varepsilon_N$$

ε_F: フェルミエネルギー

$$h\underbrace{(\phi_1 \cdots \phi_N)}_{\Phi} = \underbrace{(\phi_1 \cdots \phi_N)}_{\Phi} \underbrace{\begin{pmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_N \end{pmatrix}}_{\mathcal{E}}$$

$\Phi^\dagger \Phi = \Phi \Phi^\dagger = E_N$

$$h = \Phi \mathcal{E} \Phi^\dagger$$

$$H = \underbrace{c^\dagger \Phi}_{d^\dagger}\, \varepsilon\, \underbrace{\Phi^\dagger c}_{d} = \sum_{\ell=1}^{N} \varepsilon_\ell\, \underbrace{d_\ell^\dagger d_\ell}$$

$$(d^\dagger)_\ell = d_\ell$$

$$\parallel$$

$$(c^\dagger \Phi)_\ell = c^\dagger \phi_\ell$$

$$\parallel$$

$$(\quad c^\dagger \quad)\ \underbrace{\begin{pmatrix} \phi_\ell \end{pmatrix}}_{\ell}$$

$$|G\rangle = \prod_{\ell=1}^{M} d_\ell^\dagger |0\rangle = \prod_\ell (c^\dagger \phi_\ell)|0\rangle$$

$$= \left| \begin{pmatrix} \phi \end{pmatrix} \right\rangle = |\phi\rangle$$

$$\phi = \Big(\underbrace{\begin{matrix} \phi_\ell \end{matrix}}_{M} \Big)_N = (\phi_1, \cdots, \phi_M)$$

埋封状態から
つくる

$$\underline{\mathcal{A}} = \langle G | dG \rangle = \mathrm{Tr}_M A, \quad \underline{A} = \phi^\dagger d\phi$$

（$M \times M$：行列の
ベリー接続）

ゲージ変換

$h\varphi_\ell = \varphi_\ell \varepsilon_\ell$ $\quad$ $\varphi_\ell \to \varphi'_\ell = \varphi_\ell e^{i\theta_\ell}$ 自由度

$$\varphi \to \varphi' = (\varphi_1, \dots, \varphi_M) \begin{pmatrix} e^{i\theta_1} & & 0 \\ & e^{i\theta_2} & \\ & & \ddots \\ 0 & & & e^{i\theta_M} \end{pmatrix}$$

それ以外に ℓ $\quad$ $\varepsilon_{\ell_1} = \varepsilon_{\ell_2}$ $\quad$ フェルミ面から

$\ell_1 < M$ $\quad$ はなれた状態での

$\ell_2 < M$ $\quad$ 1粒子状態の縮退

φ_{ℓ_1} と φ_{ℓ_2} は一意には決まらない

(∵縮退している)

→ 一般に フェルミエネルギー以下の状態をまぜる

$\varphi \to \varphi_g = \varphi g$ とする意味

$$A = \langle G | dG \rangle = \mathrm{Tr}\, A$$

$\parallel$

$A_i dx_i$

$$A \to A_g = g^{-1} A g + g^{-1} dg$$

ゲージ変換はどうなるか？

ベリー位相は？

チャーン数は？

$$i\gamma = \int_C A = \oint_C A_i dx_i$$

$$= \int_C \mathrm{Tr}\, A \qquad \mathrm{Tr}\, A\ ?$$

$$A \to A_g = g^{-1} A g + g^{-1} dg$$

$$\operatorname{Tr} A_g = \operatorname{Tr} g^{-1} A g + \operatorname{Tr} g^{-1} dg$$

$$= \operatorname{Tr} A + \operatorname{Tr} g^{-1} dg$$

$$\operatorname{Tr} g^{-1} dg = \operatorname{Tr} d \log g = d \operatorname{Tr} \log g \sim \log \det g$$

$$\det g = e^{i \sum_\ell \lambda_\ell}$$

$$\log \det g = i \sum_\ell \lambda_\ell$$

$$\left(\lambda^{-1} d\lambda = d \log \lambda \right)$$

$$g \sim \begin{pmatrix} e^{i\lambda_1} & & \\ & \ddots & \\ & & e^{i\lambda_M} \end{pmatrix}$$

$$\log g \sim \begin{pmatrix} i\lambda_1 & & \\ & \ddots & \\ & & i\lambda_M \end{pmatrix}$$

$$\operatorname{Tr} \log g = \sum i \lambda_\ell = \log \det g$$

$$\operatorname{Tr} A_g = \operatorname{Tr} A + d \log \det g$$

$$i\gamma_g = \int A_g = \int \left(A + d\log\det g\right) \qquad \det g = \Delta$$

$$= |\Delta| e^{i\operatorname{Arg}\Delta}$$

$$= i\gamma + \oint d\log \cancel{\det g}\,\Delta$$

$$= i\gamma + i\oint d\operatorname{Arg}\Delta$$

$$\gamma_g = \gamma + \underbrace{\oint d\operatorname{Arg}\Delta}_{2\pi n} \qquad n: \text{整数}$$

$$\gamma_g \equiv \gamma, \quad (\mathrm{mod}\ 2\pi)$$

Chern数は?

gauge不変な表式

$$C = \frac{1}{2\pi i}\int d^2 S \cdot \mathrm{Tr}P(\nabla\Psi \times \nabla\Psi)$$

$$\rightarrow \ = \frac{1}{2\pi i}\int \mathrm{Tr}\, P\, dP\, dP \qquad P: |\psi\rangle \text{への projection}$$

$$\rightarrow \quad \Psi = (|\psi_1\rangle, \cdots, |\psi_M\rangle) \leftarrow$$

$\langle \psi_\ell | \psi_{\ell'} \rangle = \delta_{\ell\ell'}$

$$P = \sum_{\ell=1}^{M} P_\ell \quad , \ P_\ell = |\psi_\ell\rangle\langle\psi_\ell| \quad , \ P^2 = P \leftarrow$$

$$= |\psi_1\rangle\langle\psi_1| + \cdots + |\psi_M\rangle\langle\psi_M|$$

$$= (|\psi_1\rangle, \cdots, |\psi_M\rangle)\begin{pmatrix}\langle\psi_1| \\ \vdots \\ \langle\psi_M|\end{pmatrix} = \Psi\Psi^\dagger$$

$$C = \frac{1}{2\pi i}\int \mathrm{Tr}_M\, P\, dP\, dP\, P \quad , P = \psi\psi^\dagger$$

gauge 不変

ベリー接続?

$$\mathrm{Tr}\, \underline{P\, dP^2}\, P$$

$$A = \psi^\dagger d\psi \quad , \psi^\dagger\psi = 1$$

$$d\psi^\dagger\psi + \psi^\dagger d\psi = 0$$

$$dP = \overset{①}{d\psi\psi^\dagger} + \overset{②}{\psi d\psi^\dagger}$$

$$dP^2 = d\psi\psi^\dagger d\psi\psi^\dagger + d\psi\psi^\dagger\psi d\psi^\dagger + \psi d\psi^\dagger d\psi\psi^\dagger + \psi d\psi^\dagger\psi d\psi^\dagger$$

①×① ①×② ②① ②②

$$P\, dP^2 = \psi\underbrace{\psi^\dagger d\psi}_{A}\underbrace{\psi^\dagger d\psi}_{A}\psi^\dagger + \psi\psi^\dagger d\psi\underbrace{\psi^\dagger\psi}_{1} d\psi^\dagger$$

$$+ \psi\underbrace{\psi^\dagger\psi}_{1} d\psi^\dagger d\psi\psi^\dagger + \psi\underbrace{\psi^\dagger\psi}_{1} d\psi^\dagger\psi d\psi^\dagger$$

$$= \psi A^2\psi^\dagger + \psi\psi^\dagger d\psi\, d\psi^\dagger + \psi\, d\psi^\dagger d\psi\psi^\dagger + \psi\, d\psi^\dagger\psi\, d\psi^\dagger$$

$$PdP^2 = \psi A^2 \psi^\dagger + \psi\psi^\dagger d\psi\, d\psi^\dagger + \psi\, d\psi^\dagger d\psi\, \psi^\dagger + \psi\, d\psi^\dagger \psi\, d\psi^\dagger$$

$$PdP^2P = \psi A^2 \underbrace{\psi^\dagger \psi}_{=1} \psi^\dagger + \psi \overbrace{\psi^\dagger d\psi}^{A} \underbrace{d\psi^\dagger \psi}_{-\psi^\dagger d\psi} \psi^\dagger$$

$$+ \psi\, d\psi^\dagger d\psi \underbrace{\psi^\dagger \psi}_{\substack{= \\ 1}} \psi^\dagger + \psi\, d\underbrace{\psi^\dagger \psi}\, d\underbrace{\psi^\dagger \psi}\, \psi^\dagger$$

$$+ \psi A^2 \psi^\dagger$$

$$= \psi A^2 \psi^\dagger - \psi A^2 \psi^\dagger + \psi\, d\psi^\dagger d\psi\, \psi^\dagger + \psi A^2 \psi^\dagger$$

$$= \psi \left(A^2 \underbrace{+ d\psi^\dagger d\psi}_{F} \right) \psi^\dagger \qquad F = d\psi^\dagger d\psi + A^2$$

$$= \psi F \psi^\dagger$$

$$C = \frac{1}{2\pi i}\int \mathrm{Tr}\, P\,dP^2 = \frac{1}{2\pi i}\int \mathrm{Tr}\, \psi F \psi^\dagger$$

$$A = \psi^\dagger d\psi$$
$$dA = d\psi^\dagger d\psi$$

$$= \frac{1}{2\pi i}\int \mathrm{Tr}\, F$$

Chern数
非可換版

$$F = \overbrace{d\psi^\dagger d\psi}^{dA} + A^2 = d\psi^\dagger d\psi + \psi^\dagger d\psi\, \psi^\dagger d\psi$$

$$= dA + A^2 = d(dx_i A_i) + dx_i A_i\, dx_j A_j$$

$$dx_i\, dx_j\, \partial_j A_i$$

$$= dx_i\, dx_j\, (\partial_j A_i + A_i A_j)$$

$$= \sum_{i<j} dx_i\, dx_j\, (\partial_j A_i - \partial_i A_j + A_i A_j - A_j A_i)$$

$$[A_i, A_j]$$
一成分なら0

$$= \sum_{i<j} dx_i\, dx_j\, F_{ij}$$

gauge 不変性 P で書けているから自明

F はどう変換するか？

$F = dA + A^2$, $F_g = dA_g + A_g^2$

$A_g = g^{-1} A g + g^{-1} dg$

gauge 理論から

Berry接続の議論

$\underline{\underline{\Psi}}$ $\quad \Psi_g = \Psi g$ $\quad d\Psi_g = d\Psi g + \Psi dg$, ✓

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$\frac{dx}{x} = -\frac{1}{x} dx \frac{1}{x}$

$F_g = dA_g + A_g^2 = d\Psi_g^\dagger d\Psi_g + (g^{-1} A g + g^{-1} dg)^2$

$d\Psi^\dagger = g^\dagger d\Psi^\dagger + \underline{\underline{dg^\dagger}} \Psi^\dagger$ $\quad dg^\dagger$?

$\quad\quad \parallel \quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

$g^{-1} d\Psi^\dagger - g^{-1} dg g^{-1} \Psi^\dagger$ $\quad dg^{-1}$

$g^{-1} g = 1$, $dg^{-1} g + g^{-1} dg = 0$

$\underline{\underline{dg^{-1} = -g^{-1} dg g^{-1}}}$

$$F_g = d\psi_g^{\dagger} d\psi_g + (g^{-1}Ag + g^{-1}dg)^2$$

$\psi_g = \psi g$, $d\psi_g = d\psi g + \psi dg$

$d\psi^{\dagger} = g^{\dagger} d\psi^{\dagger} - g^{-1} dg g^{-1} \psi^{\dagger}$

$$= (g^{-1} d\psi^{\dagger} - g^{-1} dg\, g^{-1} \psi^{\dagger})(d\psi g + \psi dg)$$

$$+ (g^{-1}A g + g^{-1} dg)^2$$

$$= g^{-1}(\underset{①}{d\psi^{\dagger}} - \underset{②}{dg\, g^{-1}\psi^{\dagger}})(\underset{③}{d\psi g} + \underset{④}{\psi dg}) + (\underset{Ⓐ}{g^{-1}Ag} + \underset{Ⓑ}{g^{-1}dg})^2$$

$$= \underset{①③}{g^{-1} d\psi^{\dagger} d\psi g} + \underset{①④}{g^{-1} d\psi^{\dagger} \psi dg} - \underset{②③}{g^{-1} dg\, g^{-1} \psi^{\dagger} \cdot d\psi g} - \underset{②④}{g^{-1} dg\, g^{-1} \psi^{\dagger} \psi dg}$$

$$+ \underset{Ⓐ^2}{g^{-1}A^2 g} + \underset{ⒶⒷ}{g^{-1}A\, dg} + \underset{ⒷⒶ}{g^{-1}dg\, g^{-1}Ag} + \underset{Ⓑ^2}{(g^{-1}dg)^2}$$

$$F_g = g^{-1}d\psi^\dagger d\psi g + g^{-1}\overbrace{d\psi^\dagger \psi}^{-A} dg - g^{-1}dg\, g^{-1}\overbrace{\psi^\dagger \cdot d\psi}^{A} g - g^{-1}dg\, g^{-1}\overbrace{\psi^\dagger \psi}^{1} dg$$

$$① ③ \qquad ① ④ \qquad ② ③ \qquad ② ④$$

$$+ g^{-1}A^2 g + g^{-1}A\,dg + g^{-1}dg\, g^{-1}Ag + (g^{-1}dg)^2 \qquad -(g^{-1}dg)^2$$

$$(A)^2 \qquad (A)(B) \qquad (B)(A) \qquad (B)^2 \qquad 0$$

$$= g^{-1}d\psi^\dagger d\psi g - g^{-1}A\,dg - g^{-1}dg\, g^{-1}Ag$$

$$+ g^{-1}A^2 g + g^{-1}A\,dg + g^{-1}dg\, g^{-1}Ag$$

$$= g^{-1}(\underbrace{d\psi^\dagger d\psi + A^2}_{F})g = g^{-1}Fg$$

$\mathrm{Tr}\,F_g = \mathrm{Tr}\,F$ gauge不变

$$C = \frac{1}{2\pi i}\int \mathrm{Tr}\, P\, dP^2 = \frac{1}{2\pi i}\int \mathrm{Tr}\, F \quad , \quad F = dA + A^2$$

$$= d\psi^\dagger d\psi + A^2$$

$$= \frac{1}{2\pi i}\int \mathrm{Tr}\, F_g$$

$$F_g = dA_g + A_g^2$$

$$= g^{-1} F g$$

$$A_g = g^{-1} A g + g^{-1} dg$$

(2011)

Hatsumo

NJP

$$P\, dP^2 P = \psi F \psi^\dagger$$

$$(P\, dP^2 P)^n = \psi F \underbrace{\psi^\dagger \cdot \psi}_{1} F \psi^\dagger \cdots = \psi F^n \psi^\dagger$$

$$\mathrm{Tr}\,(P\, dP^2)^n = \mathrm{Tr}\, F^n$$

$$C_n = * \int \mathrm{Tr}\, F^n = * \int \mathrm{Tr}\,(P\,dP^2)^n$$

n 次チャーン類 : gauge 不変.