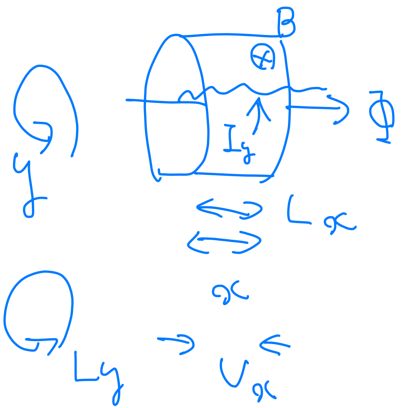
12/14 Φ Laughlin の議論



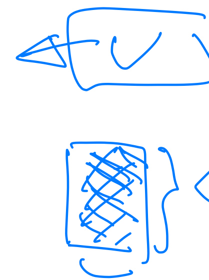
$I_y \sim V_x \qquad I_y = \sigma_{yx} V_x$

$\sigma_{yx} = \dfrac{e^2}{h} \times$ 整数 //

$H = \sum_i \frac{1}{2m}(\vec{p}_i - e\vec{A}_i)^2$

$I_y = e\left(\dfrac{N}{L_x L_y}\right) \underbrace{L_x \langle v_y \rangle}_{1[s]}$

〜 密度



$\langle v_x \rangle\, 1$

L_x

$= e\dfrac{N}{L_y}\langle v_y \rangle$

$\sum_i v_i = \sum_i \dfrac{p_i - eA_i}{m}$

$\oint A_y\, dl = -\Phi$

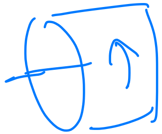
$= e\dfrac{\cancel{N}}{L_y}\underbrace{\dfrac{1}{\cancel{N}}\sum_i^N \langle v_i^y \rangle}$

PRL 7, 46 (1961) $= \dfrac{\partial H}{\partial A_y}\Big/(-e)$

$A_y L_y = -\Phi \qquad = \langle G| \dfrac{e}{L_y}\dfrac{\partial H}{\partial A_y}\Big/(-e)|G\rangle$

Byers-Yang

$-\dfrac{\partial}{\partial \Phi} = L_y \dfrac{\partial}{\partial A_y} \qquad = -\dfrac{1}{L_y}\langle G| \dfrac{\partial H}{\partial A_y}|G\rangle = \langle G|\dfrac{\partial H}{\partial \Phi}|G\rangle$

 $\to \Phi$

$$I_y = \langle G | \frac{\partial H}{\partial \Phi} | G \rangle$$

Φ の変化を断熱的に

$$※\ i\hbar \partial_t |G\rangle = H|G\rangle$$

Feynman

$$= \frac{\partial}{\partial \Phi} \langle g | H | g \rangle$$

$$H|g\rangle = |g\rangle E_\Phi$$

$$|G\rangle \neq |g\rangle$$

$$\underline{|G\rangle \propto |g\rangle}\ .$$

断熱近似定理

$$\langle \partial_\Phi g | H | g \rangle + \underline{\langle g | \partial H | g \rangle} + \langle g | H | \partial g \rangle$$

$$\langle g | \partial_\Phi H | g \rangle$$

$$\shortparallel$$

$$\langle G | \partial_\Phi H | G \rangle$$

$$|G\rangle \propto |g\rangle$$

$$\partial_\Phi \underset{\substack{\shortparallel \\ 1}}{\langle g | g \rangle} = 0$$

$$I_y = \frac{\partial E}{\partial \Phi}$$

Y. Hatsugai, Dec.14 (2020)

Laughlinの議論

Φ

$A_y L_y = \Phi$

不変

$\to V_x \leftarrow$

L_y 十分大

$\Delta\Phi = \Phi_0$ 変化

$\Delta A_y = \dfrac{\Phi_0}{L_y}$ 小さい

$$I_y \simeq \frac{\Delta E}{\Delta \Phi} = \frac{\Delta E}{\Phi_0}$$

$\Delta\Phi = \Phi_0 \Rightarrow$ $\boxed{\text{ゲージ不変性から系は不変のはず}}$

Φ: $0 \to \Phi_0$ で何が変化し得るか？

ΔE? エッジの左→右へ状態が通りすぎたと考える

n コ 整数

エッジ状態 (Halperin) より全体

$$\Delta E = neV_x$$

$$I_y = \frac{neV_x}{\Phi_0} = \frac{neV_x}{h/e} = \frac{e^2}{h} n V_x = \sigma_{yx} V_x$$

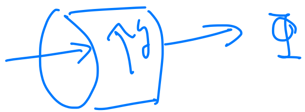
$$\sigma_{yx} = \frac{e^2}{h} n$$

整数量子 // ホール効果

$\Phi = n\Phi_0$

ψ' は周期的

ゲージ不変性



$$\oint_y \vec{A}\cdot d\vec{l} = -\Phi \quad \text{AB flux}$$

$$\oint \vec{A}'\cdot d\vec{l} = 0$$

$$\vec{A}' = 0$$

$$\boxed{\vec{A} = \vec{A}' + \nabla\chi \quad \text{ゲージ変換}}$$

$\vec{A}'$ で考えてよい。

$$\oint \nabla\chi = -\Phi.$$

$$\chi = -\frac{\Phi}{L_y}y$$

$$\vec{B} = \mathrm{rot}\vec{A} = \mathrm{rot}\vec{A}' + \underbrace{\mathrm{rot}\,\nabla\chi}_{=0} \quad \text{不変}$$

$$\frac{1}{2m}(-i\hbar\vec{\nabla} - e\vec{A})^2\psi = \frac{1}{2m}(-i\hbar\vec{\nabla} - e\vec{A}' - e\vec{\nabla}\chi)^2\psi$$

$$(\psi = e^{i\theta}\psi')$$

$$\theta = i\chi$$

$$\hbar\nabla\theta - e\nabla\chi = 0$$

$$\theta = \frac{e}{\hbar}\chi$$

$$\frac{1}{2m}(p - eA)^2\psi = E\psi$$

$$\frac{1}{2m}(p)^2\psi' = E\psi'$$

$$= \frac{1}{2m}(-i\hbar\vec{\nabla} - i\hbar i\nabla\theta - e\underbrace{\vec{A}'}_{0} - e\nabla\chi)\psi' = E\psi$$

$$= E e^{i\theta}\psi'$$

$$\psi' = e^{-i\theta}\psi = e^{-i\frac{e}{\hbar}\chi}\psi$$

$$= e^{+i2\pi\frac{e}{h}\frac{\Phi}{L_y}y}\psi$$

$$e^{i\theta}\frac{1}{2m}(-i\hbar\vec{\nabla})^2\psi' = E\psi'$$

もし $\psi(y) = \psi(y+L_y)$

$$\psi'(y+L_y) = e^{i2\pi\frac{\Phi}{\Phi_0}}\psi'(y)$$

☆ ゲージ固定とベリー接続の計算 JPSJ 73, 2604 (2004) Y. Hatsugai

問題 $H(x=\varphi) = \begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}$ 2×2 ?

$$H|\psi\rangle = |\psi\rangle E \qquad A = \langle\psi|d\psi\rangle = \underbrace{A_i\, dx_i}_{} = A_\varphi d\varphi$$

$$A_i = \langle\psi|\partial_i\psi\rangle$$

$$A_\varphi = \langle\psi|\partial_\varphi|\psi\rangle$$

$$\gamma = i\oint_C A = i\int_0^{2\pi} d\varphi\, A_\varphi = i\int_0^{2\pi} d\varphi\, \langle\psi|\partial_\varphi\psi\rangle$$

ベリー位相

$$H|\psi\rangle = |\psi\rangle E$$

$$\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} E, \quad \det\left(E - \begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}\right) \cdots$$

$$= E^2 - 1 = 0, \quad E = \pm 1$$

$E = +1$ $\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$, $a = 1 \Rightarrow b = ?$ // ...

位相の不定性

$H|\psi\rangle = |\psi\rangle$　　$H\underbrace{|\psi\rangle e^{i\theta}} = \underbrace{|\psi\rangle e^{i\theta}}$

$|\psi'\rangle$　　$H|\psi'\rangle = |\psi'\rangle$

$|\psi\rangle$ の位相は不定（計算できない）

→ 位相を決める（$a=1$ としてみる）

一般に　$P = |\psi\rangle\langle\psi|$　：位相によらない　$|\psi'\rangle = |\psi\rangle e^{i\theta}$

⇙ 射影 (projection)　$|\psi'\rangle\langle\psi'| = |\psi\rangle \underbrace{e^{i\theta} e^{-i\theta}}_{1} \langle\psi|$

$P^2 = |\psi\rangle \overbrace{\langle\psi|\psi\rangle}^{1} \langle\psi|$

$= |\psi\rangle\langle\psi| = P$　　$P' = P$　gauge 不変　　$= |\psi\rangle\langle\psi|$

ゲージ固定の方法　$|\phi\rangle$ 任意に1つ固定

$$|\psi_\phi\rangle \equiv P|\phi\rangle \quad \text{一意に決まる（規格化されていない）}$$

$|\langle\phi|\psi\rangle|^2$

$$N_\phi = \langle\psi_\phi|\psi_\phi\rangle = \langle\phi|PP|\phi\rangle = \langle\phi|P^2|\phi\rangle = \langle\phi|P|\phi\rangle = \langle\phi|\psi\rangle\langle\psi|\phi\rangle$$

Y. Hatsugai, Dec.14 (2020)

$$|\psi_\phi\rangle = P|\phi\rangle\ , \quad N_\phi = \langle\psi_\phi|\psi_\phi\rangle = |\langle\phi|\psi\rangle|^2$$

$$|\tilde{\psi}\rangle = |\psi_\phi\rangle/\sqrt{N} \qquad \underline{\underline{(N\neq 0)}}$$

$$A = \langle\tilde{\psi}|d\tilde{\psi}\rangle \Rightarrow \gamma = i\oint A$$

<u>ex)</u> $H = \begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}$, $|\phi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

例えば $\begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ e^{-i\varphi} \end{pmatrix} = |\psi\rangle$

$$P = |\psi\rangle\langle\psi| = \begin{pmatrix} 1 \\ e^{-i\varphi} \end{pmatrix}(1, e^{i\varphi}) = \begin{pmatrix} 1 & e^{i\varphi} \\ e^{-i\varphi} & 1 \end{pmatrix} *$$

gauge 不変

$$|\psi_\phi\rangle = P|\phi\rangle = (*)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ e^{-i\varphi} \end{pmatrix}$$

$$N = \langle\psi_\phi|\psi_\phi\rangle = 2 \longrightarrow |\psi_\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{-i\varphi} \end{pmatrix}$$

$$|\phi'\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |\psi_{\phi'}\rangle = P\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{i\varphi} \\ e^{-i\varphi} & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\varphi} \\ 1 \end{pmatrix}$$

$$N_{\phi'} = \langle \psi_{\phi'} | \psi_{\phi'} \rangle = 2, \quad |\tilde{\psi}_{\phi'}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\varphi} \\ 1 \end{pmatrix}$$

まとめると

$$|\phi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to |\tilde{\psi}_\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{-i\varphi} \end{pmatrix} \quad // \quad \checkmark$$

$$|\phi'\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to |\tilde{\psi}_{\phi'}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\varphi} \\ 1 \end{pmatrix} = |\psi_\phi\rangle e^{i\varphi}$$

$$A_\phi = \langle \tilde{\psi}_\phi | d\tilde{\psi}_\phi \rangle = d\varphi A_\varphi$$

$$A_\varphi = \langle \tilde{\psi}_\phi | \partial_\varphi \tilde{\psi}_\phi \rangle = \frac{1}{\sqrt{2}}\left(1, e^{i\varphi}\right)\begin{pmatrix} 0 \\ -i e^{-i\varphi} \end{pmatrix} \Big/ \sqrt{2} = \frac{1}{2}(-i) = -\frac{1}{2}i$$

$$\nu_\phi = i\int_0^{2\pi} d\varphi A_\varphi = 2\pi\left(-\frac{1}{2}i^2\right) = \pi \quad //$$

$$|\tilde{\psi}_{\phi'}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\varphi} \\ 1 \end{pmatrix}$$

$$A_{\phi'} = \langle\tilde{\psi}_{\phi'}|d\tilde{\psi}_{\phi'}\rangle = d\varphi\, A'_{\varphi}$$

$$A'_{\varphi} = \langle\tilde{\psi}_{\phi'}|\partial_{\varphi}\tilde{\psi}_{\phi'}\rangle = \frac{1}{\sqrt{2}}(e^{-i\varphi},1)\begin{pmatrix} i e^{i\varphi} \\ 0 \end{pmatrix}/\sqrt{2}$$

$$= \frac{1}{2}i$$

$$\gamma_{\phi'} = i\int_0^{2\pi} A'_{\varphi}\, d\varphi = i2\pi\frac{1}{2}i = -\pi$$

$$\gamma_{\phi} = \pi$$

$$\gamma_{\phi'} = -\pi$$

$$\gamma_{\phi} - \gamma_{\phi'} = 2\pi$$

$$\gamma_{\phi} \equiv \gamma_{\phi'} \equiv \pi \pmod{2\pi}$$

(4)

$\big[\quad\big]\big[\psi\big] = \big[\ \big] E \qquad x = x_0 \qquad \psi(x_0)$

gauge ?

$\big[\quad\big]\big[\psi\big] = \big[\ \big] E \qquad x' = x_0 + \Delta x \qquad \psi(x')$

$\psi(\ ) = \psi(\ ) e^{i\theta},$

$P(x), \quad P(x')$

$\begin{pmatrix} a_1 \\ \\ \end{pmatrix} \qquad a \geq 0$

§ Chern数の量子化　Kohmoto '85

$$C = \frac{1}{2\pi i}\int_{T^2} d^2x\,(\mathrm{rot}\,\vec{A})_z = \frac{1}{2\pi i}\int_{T^2} d^2x\,(\partial_x A_y - \partial_y A_x)$$

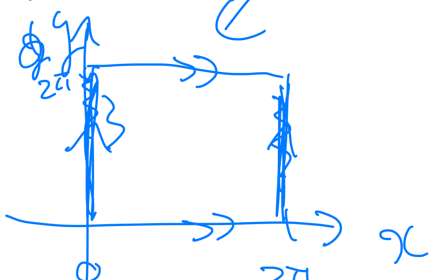
T^2 = 境界がない

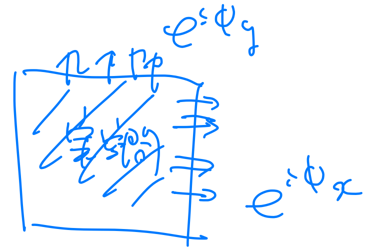
$$A_x = \langle \psi | \partial_x \psi \rangle$$

$$A_y = \langle \psi | \partial_y \psi \rangle$$

$$\begin{cases} x = \phi_x \\ y = \phi_y \end{cases}$$

(パラメータ空間

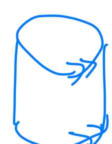


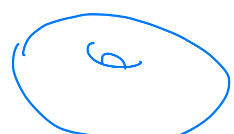


$$H|\psi\rangle = |\psi\rangle E$$

$$|\psi(x,y)\rangle$$

$$\boxed{e^{i0} = e^{i2\pi}}$$



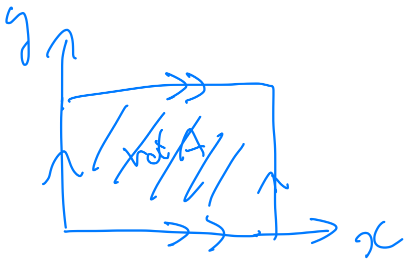


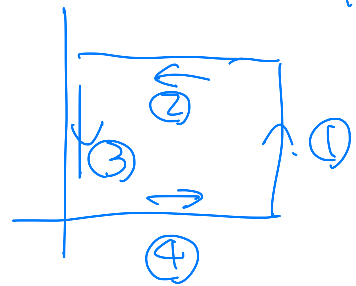
トーラス T^2

Y. Hatsugai, Dec.14 (2020)

$$C = \frac{1}{2\pi i}\int_{T^2} d^2x \, (\mathrm{rot}\,\vec{A})_z = \frac{1}{2\pi i}\int_{\partial T^2} \vec{A}\cdot d\vec{l} = 0$$

$$\int_{\partial T^2} = \int_{①} + \int_{②} + \int_{③} + \int_{④} = 0$$





$$\int_{①} = -\int_{③}$$

$$\int_{②} = -\int_{④}$$

Chern数は常に零?

計算にはゲージ固定が必要

大局的に1つのゲージでゲージ固定できれば $\Rightarrow C=0$

$C \neq 0 \Rightarrow$ 大局的なゲージは存在しないはず

ゲージ固定できない場所がある（規格化できない！） Kohmoto 85

ある状態 $|\phi\rangle$ をとってくる

P : gauge不変　$|\psi_\phi\rangle = P|\phi\rangle$

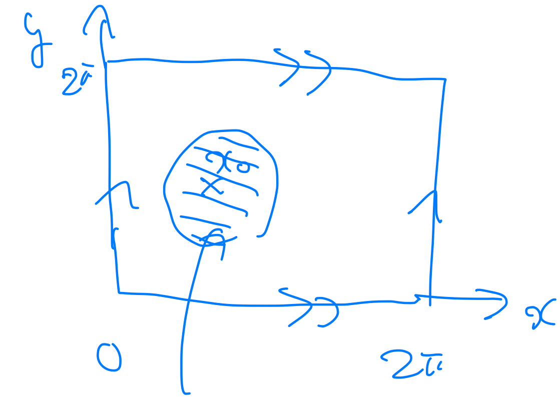
$$|\tilde{\psi}_\phi\rangle = |\psi_\phi\rangle / \sqrt{N_\phi},$$

$$N_\phi = \langle\psi_\phi|\psi_\phi\rangle = \langle\phi|P|\phi\rangle = |\langle\phi|\psi\rangle|^2$$

✓ $\boxed{N_\phi \neq 0 \text{ for all } x \in T^2}$ $|\phi\rangle$!　$|\tilde{\psi}_\phi\rangle \to A_{\tilde{\phi}}$

$\to C = 0$

ex (3) $x = x_0$ $N_\phi = |\langle \phi | \psi \rangle|^2 = 0$



$N_\phi = 0$

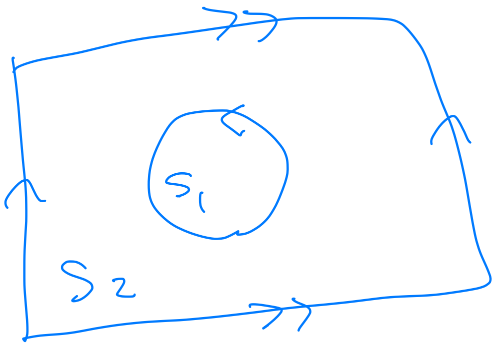
(ii) S_1, $S_2 = T^2 \setminus S_1$ (トーラス上で S_1 以外)

$x \in S_1$ $|\tilde{\psi}_{\phi'}\rangle = P|\phi'\rangle / \sqrt{N_{\phi'}}$ $|\phi'\rangle \neq |\phi\rangle$

$$\boxed{\begin{array}{l} N_{\phi'} = |\langle \phi' | \psi \rangle|^2 \neq 0 \\ x \in S_1 \end{array}}$$

$x \in S_2 \Rightarrow |\tilde{\psi}_\phi\rangle = P|\phi\rangle / \sqrt{N_\phi}$

patch work ぬいわける



$$C = \frac{1}{2\pi i}\left\{\int_{S_1} d^2x\ \mathrm{rot}A_{\phi'} + \int_{S_2} d^2x\ \mathrm{rot}A_{\phi}\right\}$$

$$= \frac{1}{2\pi i}\left\{\int_{\partial S_1} d\vec{l}\cdot\vec{A}_{\phi} + \int_{\underbrace{\partial S_2}_{-\partial S_1}} d\vec{l}\cdot\vec{A}_{\phi'}\right\} = *$$

Kohmoto 85

$\partial S_1 = -\partial S_2$

境界の向きは逆

$$= \frac{1}{2\pi i}\int_{\partial S_1} d\vec{l}\,(A_{\phi} - A_{\phi'})$$

$x \in \partial S_1$ $\quad (\tilde{\psi}_{\phi}) = (\tilde{\psi}_{\phi'})e^{i\theta} \longrightarrow A_{\phi} = \langle\tilde{\psi}_{\phi}|d\tilde{\psi}_{\phi}\rangle$ "$A_{\phi'} + i\,d\theta$"

境界上では2つのゲージ固定は両立

$A_{\phi'} = \langle\tilde{\psi}_{\phi'}|d\tilde{\psi}_{\phi'}\rangle$

$$C = \frac{1}{2\pi i}\int_{\partial S_1} d(i\,d\theta) = \frac{1}{2\pi}\int_{\partial S_1} d\theta = \frac{1}{2\pi}2\pi n = n \in \mathbb{Z}$$

= Chern数の量子化

$e^{i\theta}$は∂S_1上で一価