

2018/08/34

量子力学第19回

相原 大地

 $j = 1/2$  の回転行列

$$R(\alpha, \beta, \gamma) |j, m\rangle = |j, m'\rangle [D^j(R(\alpha, \beta, \gamma))]_{m'm}$$

$$\langle j, m' | R | j, m \rangle = \langle j, m' | e^{-iJ_z \alpha} e^{-iJ_y \beta} e^{-iJ_z \gamma} | j, m \rangle$$

$$= e^{-im'\alpha} \langle j, m' | e^{-iJ_y \beta} | j, m \rangle e^{-im\gamma}$$

$$= [D^j(R(\alpha, \beta, \gamma))]_{m'm}$$

$$[D^j(R(\alpha, \beta, \gamma))]_{m'm} = e^{-im'\alpha} d_{m'm}^j e^{-im\gamma}$$

$$d_{m'm}^j = \langle j, m' | e^{-iJ_y \beta} | j, m \rangle$$

$$d_{m'm}^j = \sum_R (-1)^{R-m+m'} \frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-R)! R! (R-m+m')! (j-R-m')!}$$

$$\boxed{\begin{matrix} ( )! の ( )! の \\ 0! 以上の和 \end{matrix}} \times \cos^{2j-2R+m-m'} \frac{\beta}{2} \sin^{2R-m+m'} \frac{\beta}{2}$$

 $j = 1/2$  のときは

$$m' = m = 1/2$$

$$d_{1/2, 1/2}^{1/2} = \sum_{R=0}^0 (-1)^R \frac{\sqrt{(\frac{1}{2}+\frac{1}{2})!(\frac{1}{2}-\frac{1}{2})!(\frac{1}{2}+\frac{1}{2})!(\frac{1}{2}-\frac{1}{2})!}}{(\frac{1}{2}+\frac{1}{2}-R)! R! (R-\frac{1}{2}+\frac{1}{2})! (\frac{1}{2}-R-\frac{1}{2})!} (-R)! \times \cos^{1-2R+\frac{1}{2}-\frac{1}{2}} \frac{\beta}{2} \sin^{2R-\frac{1}{2}+\frac{1}{2}} \frac{\beta}{2}$$

$$= \cos \frac{\beta}{2}$$

$$m' = 1/2, m = -1/2$$

$$d_{1/2, -1/2}^{1/2} = \sum_{R=0}^0 (-1)^{R+1} \frac{\sqrt{(\frac{1}{2}-\frac{1}{2})!(\frac{1}{2}+\frac{1}{2})!(\frac{1}{2}+\frac{1}{2})!(\frac{1}{2}-\frac{1}{2})!}}{(\frac{1}{2}-\frac{1}{2}-R)! R! (R+\frac{1}{2}+\frac{1}{2})! (\frac{1}{2}-R-\frac{1}{2})!} (-R)! \times \cos^{1-2R-\frac{1}{2}-\frac{1}{2}} \frac{\beta}{2} \sin^{2R+\frac{1}{2}+\frac{1}{2}} \frac{\beta}{2}$$

$$= -\sin \frac{\beta}{2}$$

$$m' = -1/2, m = 1/2$$

$$d_{-1/2, 1/2}^{1/2} = \sum_{R=1}^1 (-1)^{R-1} \frac{\sqrt{(\frac{1}{2}+\frac{1}{2})!(\frac{1}{2}-\frac{1}{2})!(\frac{1}{2}-\frac{1}{2})!(\frac{1}{2}+\frac{1}{2})!}}{(\frac{1}{2}+\frac{1}{2}-R)! R! (R-\frac{1}{2}-\frac{1}{2})! (\frac{1}{2}-R+\frac{1}{2})!} (-R)! \times \cos^{1-2R+\frac{1}{2}+\frac{1}{2}} \frac{\beta}{2} \sin^{2R-\frac{1}{2}-\frac{1}{2}} \frac{\beta}{2}$$

$$= \sin \frac{\beta}{2}$$

$$m' = m = -1/2$$

$$d_{-1/2, -1/2}^{1/2} = \sum_{R=0}^0 (-1)^R \frac{\sqrt{(\frac{1}{2}-\frac{1}{2})!(\frac{1}{2}+\frac{1}{2})!(\frac{1}{2}-\frac{1}{2})!(\frac{1}{2}+\frac{1}{2})!}}{(\frac{1}{2}-\frac{1}{2}-R)! R! (R+\frac{1}{2}-\frac{1}{2})! (\frac{1}{2}-R+\frac{1}{2})!} (-R)! \times \cos^{1-2R-\frac{1}{2}+\frac{1}{2}} \frac{\beta}{2} \sin^{2R+\frac{1}{2}-\frac{1}{2}} \frac{\beta}{2}$$

$$= \cos \frac{\beta}{2}$$

$$\begin{aligned}
 d^{1/2} &= \begin{pmatrix} \cos \frac{\beta}{2} & -\sin \frac{\beta}{2} \\ \sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{pmatrix} \begin{matrix} m' = 1/2 \uparrow \\ m' = -1/2 \downarrow \end{matrix} \\
 D^{1/2} &= \begin{pmatrix} e^{-i\frac{1}{2}\alpha} & 0 \\ 0 & e^{i\frac{1}{2}\alpha} \end{pmatrix} \begin{pmatrix} \cos \frac{\beta}{2} & -\sin \frac{\beta}{2} \\ \sin \frac{\beta}{2} & \cos \frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{-i\frac{1}{2}\alpha} & 0 \\ 0 & e^{i\frac{1}{2}\alpha} \end{pmatrix} \\
 &= \begin{pmatrix} e^{-i\frac{1}{2}(\alpha+\beta)} \cos \frac{\beta}{2} & -e^{-i\frac{1}{2}(\alpha-\beta)} \sin \frac{\beta}{2} \\ e^{i\frac{1}{2}(\alpha-\beta)} \sin \frac{\beta}{2} & e^{i\frac{1}{2}(\alpha+\beta)} \cos \frac{\beta}{2} \end{pmatrix} \\
 &= e^{-i\frac{\alpha}{2} \sigma_z} e^{-i\frac{\beta}{2} \sigma_y} e^{-i\frac{\alpha}{2} \sigma_z}
 \end{aligned}$$

Unitary 表現 であるから.

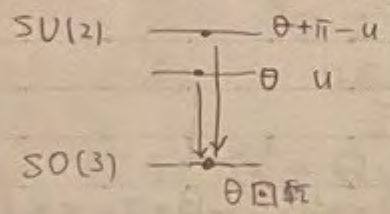
$$[D^{1/2}]^\dagger = [D^{1/2}]^\dagger \quad \det D^{1/2} = \cos^2 \frac{\beta}{2} - (-\sin^2 \frac{\beta}{2}) = 1 \quad \Rightarrow D^{1/2} \in SU(2)$$

$\vec{n}$  ( $|\vec{n}|=1$ ) 軸周りの  $\theta$  回転

$$* e^{-i\frac{\theta}{2} \vec{n} \cdot \vec{\sigma}} = E_2 \cos \frac{\theta}{2} - i \vec{n} \cdot \vec{\sigma} \sin \frac{\theta}{2} \in SU(2)$$

同じ回転を2種類に表す  $\theta + 2\pi$  回転

$$\begin{aligned}
 e^{-i\frac{\theta+2\pi}{2} \vec{n} \cdot \vec{\sigma}} &= E_2 \cos(\frac{\theta}{2} + \pi) - i \vec{n} \cdot \vec{\sigma} \sin(\frac{\theta}{2} + \pi) \\
 &= -e^{-i\frac{\theta}{2} \vec{n} \cdot \vec{\sigma}}
 \end{aligned}$$



$SO(3) \cong SU(2)$  の対称性は 1対2  
 $\Rightarrow \theta = 1/2$  の時の表現は2面であるという、2の表現を2面表現という  
 spin 1/2 では  $\tau$  と  $-\tau$  に戻るためには  $4\pi$  回転必要 (実験で確認)

一般に  $|\vec{n}|=1$  なるベクトル  $\vec{n}$  に対し,

$$(\vec{n} \cdot \vec{\sigma})^2 = |\vec{n}|^2 E = E \quad \leftarrow (\vec{A} \cdot \vec{\sigma})(\vec{B} \cdot \vec{\sigma}) = \vec{A} \cdot \vec{B} E + i \vec{\sigma} \cdot (\vec{A} \times \vec{B})$$

$$\vec{n} \cdot \vec{\sigma} = P_+ - P_-$$

$$P_{\pm} = \frac{1}{2} (E_2 \pm \vec{n} \cdot \vec{\sigma}) \quad \text{と書く}$$

$$\begin{aligned}
 P_{\pm}^2 &= \frac{1}{4} (E_2 \pm \vec{n} \cdot \vec{\sigma})^2 = \frac{1}{4} (E_2 \pm 2\vec{n} \cdot \vec{\sigma} + E_2) \\
 &= \frac{1}{2} (E_2 \pm \vec{n} \cdot \vec{\sigma}) = P_{\pm} = P_{\pm}^\dagger
 \end{aligned}$$

projection

$$\begin{cases}
 P_+ + P_- = E_2 \\
 P_+ P_- = \frac{1}{4} (E_2 + \vec{n} \cdot \vec{\sigma})(E_2 - \vec{n} \cdot \vec{\sigma}) = \frac{1}{4} (E_2 - E_2) = 0
 \end{cases}$$



よ-7. 関数  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  は関数に 関して

$$\begin{aligned} (\vec{n} \cdot \vec{\sigma})^2 &= (P_+ - P_-)^2 = P_+^2 - P_+ P_- - P_- P_+ + (-1)^2 P_-^2 \\ &= P_+ + (-1)^2 P_- \end{aligned}$$

$$(\vec{n} \cdot \vec{\sigma})^n = P_+ + (-1)^n P_-$$

$$\begin{aligned} \therefore f(a \vec{n} \cdot \vec{\sigma}) &= a_0 (P_+ + P_-) + a_1 (P_+ - P_-) + a_2 (P_+ + P_-) + \dots \\ &= f(a) P_+ + f(-a) P_- \end{aligned}$$

$$f(x) = e^{-i \frac{\theta}{2} x} \text{ とすれば } (a = -i \frac{\theta}{2})$$

$$e^{-i \frac{\theta}{2} \vec{n} \cdot \vec{\sigma}} = e^{-i \frac{\theta}{2}} P_+ + e^{i \frac{\theta}{2}} P_-$$

$$= e^{-i \frac{\theta}{2}} \frac{1}{2} (E_2 + \vec{n} \cdot \vec{\sigma}) + e^{i \frac{\theta}{2}} \frac{1}{2} (E_2 - \vec{n} \cdot \vec{\sigma})$$

$$= E_2 \cos \frac{\theta}{2} - i \vec{n} \cdot \vec{\sigma} \sin \frac{\theta}{2}$$

3次元球面

$$SU(2) \cong S^3 \quad \text{に } \cong \text{ して}$$

$$D^{1/2} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} e^{-i \frac{1}{2}(\alpha+\sigma)} \cos \frac{\beta}{2} & -e^{-i \frac{1}{2}(\alpha-\sigma)} \sin \frac{\beta}{2} \\ e^{i \frac{1}{2}(\alpha-\sigma)} \sin \frac{\beta}{2} & e^{i \frac{1}{2}(\alpha+\sigma)} \cos \frac{\beta}{2} \end{pmatrix}$$

$$\det D^{1/2} = 1 = |a|^2 + |b|^2$$

と書ける ( $\alpha, \sigma, \beta$  は実数) とおくと

$$a = e^{-i \frac{1}{2}(\alpha+\sigma)} \cos \frac{\beta}{2}$$

$$b = -e^{-i \frac{1}{2}(\alpha-\sigma)} \sin \frac{\beta}{2}$$

また,  $\alpha, \sigma$  の実数と  $\text{Re } \alpha = x_1, \text{Im } \alpha = x_2, \text{Re } \beta = x_3, \text{Im } \beta = x_4$  とおくと

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

とあり,  $(x_1, x_2, x_3, x_4)$  は 4次元空間内の 3次元球面  $S^3$  上を動く。

$$\text{つまり } SU(2) \cong S^3$$

これは 2次元平面内の単位円 (1次元球面)  $S^1$

$$z = e^{i\theta} = x + iy$$

$$|z|^2 = 1 = x^2 + y^2$$

が  $U(1)$  と同相なことに対応する  $U(1) \cong S^1$

$SU(2) \rightarrow SO(3)$  の対応

$j = 1/2$  の spin 表現は  $SO(3)$  の表現であったから、 $SO(3) \rightarrow SU(2)$  の対応が具体的に与えられる。すなわち、

$$U = e^{-i \frac{\theta}{2} \hat{n} \cdot \vec{\sigma}} \in SU(2), \quad \vec{\sigma}' = U \vec{\sigma} U^\dagger \quad \text{と 与える。}$$

$$(\vec{\sigma}')^\dagger = (U^\dagger)^\dagger \vec{\sigma}^\dagger U^\dagger = U \vec{\sigma} U^\dagger = \vec{\sigma}' \quad \therefore (\sigma'_\alpha)^\dagger = \sigma_\alpha$$

$$(\sigma'_\alpha)^2 = U \sigma_\alpha U^\dagger U \sigma_\alpha U^\dagger = U \sigma_\alpha^2 U^\dagger = U U^\dagger = E_2$$

$$\begin{aligned} \alpha \neq \beta \text{ のとき, } \sigma'_\alpha \sigma'_\beta &= U \sigma_\alpha U^\dagger U \sigma_\beta U^\dagger \\ &= U \sigma_\alpha \sigma_\beta U^\dagger \\ &= i U \epsilon_{\alpha\beta\gamma} \sigma_\gamma U^\dagger = i \epsilon_{\alpha\beta\gamma} \sigma'_\gamma \end{aligned}$$

$$\text{Tr } \vec{\sigma}' = \text{Tr } U \vec{\sigma} U^\dagger = \text{Tr } U^\dagger U \vec{\sigma} = \text{Tr } \vec{\sigma} = 0$$

よって、Pauli 行列の実係数の線形和として次のように展開できる

$$\sigma'_\alpha = Q_{\alpha\beta} \sigma_\beta \quad Q_{\alpha\beta} \in \mathbb{R}$$

$$\begin{aligned} \pm 5 \text{に } \{\sigma'_\alpha, \sigma'_\beta\} &= Q_{\alpha\alpha'} Q_{\beta\beta'} \{\sigma_{\alpha'}, \sigma_{\beta'}\} \\ &= Q_{\alpha\alpha'} Q_{\beta\beta'} 2\delta_{\alpha'\beta'} \\ &= 2Q_{\alpha\alpha'} Q_{\beta\beta'} \delta_{\alpha\beta} \quad \Rightarrow Q\hat{Q} = E_3 \quad Q = \text{直交} \\ &\quad \ominus 2\delta_{\alpha\beta} \end{aligned}$$

$$\pm 5 \text{に } \sigma'_1 \sigma'_2 \sigma'_3 = U \sigma_1 \sigma_2 \sigma_3 U^\dagger = i U U^\dagger = i E_2$$

$$\begin{aligned} \text{また } \sigma'_1 \sigma'_2 \sigma'_3 &= Q_{1\alpha} Q_{2\beta} Q_{3\gamma} \sigma_\alpha \sigma_\beta \sigma_\gamma \\ &= \sum_{\sigma} \left[ \sum_{\alpha \neq \beta} \underbrace{(Q_{1\alpha} Q_{2\alpha})}_{\parallel} Q_{3\sigma} \sigma_\sigma + \sum_{\alpha \neq \beta} Q_{1\alpha} Q_{2\beta} Q_{3\sigma} \sigma_\alpha \sigma_\beta \sigma_\sigma \right] \end{aligned}$$

$$(Q\hat{Q})_{12} = (E_3)_{12} = 0 \quad \beta, \sigma \text{ は } \alpha \text{ と異なる}$$

$$= \sum_{(\alpha, \beta, \gamma) = P(123)} Q_{1\alpha} Q_{2\beta} Q_{3\gamma} \sigma_\alpha \sigma_\beta \sigma_\gamma$$

$\simeq 123$  の置換 (奇数次置換)

$$= \sum_{(\alpha, \beta, \gamma) = P(123)} Q_{1\alpha} Q_{2\beta} Q_{3\gamma} i E_2 \epsilon_{\alpha\beta\gamma}$$

$$= i E_2 \det Q$$

よって、 $\det Q = 1$  直交行列かつ  $\det Q = 1$

$$SU(2) \ni U \rightarrow Q \in SO(3)$$

$$\pm U \rightarrow Q = 2 \text{対} 1$$