

例1: 具体的なクレブシュゴルダン係数

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

 $j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, n = 1$ とし、 $m = m_1 + m_2$ との基底を次のようにとる

$$\Psi_1 = \left(\left| \frac{1}{2}, \frac{1}{2} \right\rangle \right) = (|\uparrow\uparrow\rangle)$$

$$\Psi_0 = \left(\left| \frac{1}{2}, -\frac{1}{2} \right\rangle, \left| -\frac{1}{2}, \frac{1}{2} \right\rangle \right) = (|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle)$$

$$\Psi_{-1} = \left(\left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \right) = (|\downarrow\downarrow\rangle)$$

$$j=1 \begin{cases} |1, 1\rangle = |\uparrow\uparrow\rangle = \Psi_1(1) \star \\ |1, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = \Psi_0\left(\frac{1}{\sqrt{2}}\right) \star \\ |1, -1\rangle = |\downarrow\downarrow\rangle = \Psi_{-1}(1) \star \end{cases}$$

$$j=0 \rightarrow |0, 0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \Psi_0\left(\frac{1}{\sqrt{2}}\right) \star$$

これはクレブシュゴルダン係数を使えば次のようにかける

$$|1, 1\rangle = \left| \frac{1}{2}, \frac{1}{2} \right\rangle \left\langle \frac{1}{2}, \frac{1}{2} \left| 1, 1 \right\rangle \right.$$

$$|1, 0\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2} \left| 1, 0 \right\rangle \right. + \left| -\frac{1}{2}, \frac{1}{2} \right\rangle \left\langle -\frac{1}{2}, \frac{1}{2} \left| 1, 0 \right\rangle \right.$$

$$|0, 0\rangle = \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle \frac{1}{2}, -\frac{1}{2} \left| 0, 0 \right\rangle \right. + \left| -\frac{1}{2}, \frac{1}{2} \right\rangle \left\langle -\frac{1}{2}, \frac{1}{2} \left| 0, 0 \right\rangle \right.$$

$$|1, -1\rangle = \left| -\frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle -\frac{1}{2}, -\frac{1}{2} \left| 1, -1 \right\rangle \right.$$

また

表 2.1: $\langle j_1, m_1, j_2, m_2 | j, m \rangle, j_1 = 1, j_2 = \frac{1}{2}, j = \frac{1}{2}$

j	m	m_1	m_2	$\langle j_1, m_1, j_2, m_2 j, m \rangle$
1	1	$\frac{1}{2}$	$\frac{1}{2}$	1
1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
1	-1	$-\frac{1}{2}$	$-\frac{1}{2}$	1

表 2.2: $\langle j_1, m_1, j_2, m_2 | j, m \rangle, j_1 = 0, j_2 = \frac{1}{2}, j = \frac{1}{2}$

j	m	m_1	m_2	$\langle j_1, m_1, j_2, m_2 j, m \rangle$
0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{2}}{2}$
0	0	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$

$$1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$$

$$\begin{aligned} \Psi_{\frac{3}{2}} &= (|1\rangle|\frac{1}{2}\rangle) \\ m\text{-定の基底: } \Psi_{\frac{1}{2}} &= (|1\rangle|-\frac{1}{2}\rangle, |0\rangle|\frac{1}{2}\rangle) \\ \Psi_{-\frac{1}{2}} &= (|1\rangle|\frac{1}{2}\rangle, |0\rangle|-\frac{1}{2}\rangle) \\ \Psi_{-\frac{3}{2}} &= (|1\rangle|-\frac{1}{2}\rangle) \end{aligned}$$

$m = 1 + \frac{1}{2} = \frac{3}{2}$ は $(|1\rangle|\frac{1}{2}\rangle)$ のみで

$$|\frac{3}{2}\frac{3}{2}\rangle = |1\rangle|\frac{1}{2}\rangle = \Psi_{\frac{3}{2}}(1) \star$$

$J = \frac{3}{2}$ の公式より $\checkmark J_-(\frac{3}{2}\frac{3}{2}) = \sqrt{3}|\frac{3}{2}\frac{1}{2}\rangle$

$$\begin{aligned} |\frac{3}{2}\frac{1}{2}\rangle &= \frac{1}{\sqrt{3}} J_- |\frac{3}{2}\frac{3}{2}\rangle = \frac{1}{\sqrt{3}} (J_{1-} + J_{2-}) (|1\rangle|\frac{1}{2}\rangle) \\ &= \frac{1}{\sqrt{3}} [J_{1-}(|1\rangle|\frac{1}{2}\rangle) + |1\rangle J_{2-}(|\frac{1}{2}\rangle)] \\ &= \frac{1}{\sqrt{3}} [\sqrt{2}|0\rangle|\frac{1}{2}\rangle + |1\rangle|-\frac{1}{2}\rangle] \\ &= \Psi_{\frac{1}{2}} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix} \star \end{aligned}$$

続いて $\checkmark J_-(\frac{3}{2}\frac{1}{2}) = 2|\frac{3}{2}-\frac{1}{2}\rangle$

$$\begin{aligned} |\frac{3}{2}-\frac{1}{2}\rangle &= \frac{1}{2} J_- |\frac{3}{2}\frac{1}{2}\rangle = \frac{1}{2} (J_{1-} + J_{2-}) \left[\frac{\sqrt{6}}{3}|0\rangle|\frac{1}{2}\rangle + \frac{\sqrt{3}}{3}|1\rangle|-\frac{1}{2}\rangle \right] \\ &= \frac{\sqrt{6}}{6} [J_{1-}|0\rangle|\frac{1}{2}\rangle + |0\rangle J_{2-}|\frac{1}{2}\rangle] + \frac{\sqrt{3}}{6} [J_-|1\rangle|-\frac{1}{2}\rangle + |1\rangle J_{2-}|-\frac{1}{2}\rangle] \\ &= \frac{\sqrt{6}}{6} [\sqrt{2}|1\rangle|\frac{1}{2}\rangle + |0\rangle|-\frac{1}{2}\rangle] + \frac{\sqrt{3}}{6} \sqrt{2}|0\rangle|-\frac{1}{2}\rangle \\ &= \frac{\sqrt{3}}{3} |1\rangle|\frac{1}{2}\rangle + \frac{\sqrt{6}}{3} |0\rangle|-\frac{1}{2}\rangle \\ &= \Psi_{-\frac{1}{2}} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix} \star \end{aligned}$$

更に $\checkmark J_-(\frac{3}{2}-\frac{1}{2}) = \sqrt{3}|\frac{3}{2}-\frac{3}{2}\rangle$

$$\begin{aligned} |\frac{3}{2}-\frac{3}{2}\rangle &= \frac{1}{\sqrt{3}} J_- |\frac{3}{2}-\frac{1}{2}\rangle = \frac{1}{\sqrt{3}} (J_{1-} + J_{2-}) \left[\frac{\sqrt{3}}{3}|1\rangle|\frac{1}{2}\rangle + \frac{\sqrt{6}}{3}|0\rangle|-\frac{1}{2}\rangle \right] \\ &= \frac{1}{3} |1\rangle J_{2-}|\frac{1}{2}\rangle + \frac{\sqrt{2}}{3} J_{1-}|0\rangle|-\frac{1}{2}\rangle = \frac{1}{3} |1\rangle|-\frac{1}{2}\rangle + \frac{\sqrt{2}}{3} \sqrt{2}|1\rangle|-\frac{1}{2}\rangle \\ &= \Psi_{-\frac{3}{2}}(1) \star \end{aligned}$$

以上の手順をもう少し見通しよく行おう。初め、最初に基底の変換を済ませることに計算しておく。

$$J_- \Psi_{\frac{3}{2}} = (J_{1-} + J_{2-})(|1\rangle|\frac{1}{2}\rangle) = (\sqrt{2}|0\rangle|\frac{1}{2}\rangle + |1\rangle|-\frac{1}{2}\rangle) = \Psi_{\frac{1}{2}} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

$$\begin{aligned} J_- \Psi_{\frac{1}{2}} &= (J_{1-} + J_{2-})(|1\rangle|-\frac{1}{2}\rangle, |0\rangle|\frac{1}{2}\rangle) = (\sqrt{2}|0\rangle|-\frac{1}{2}\rangle, \sqrt{2}|1\rangle|\frac{1}{2}\rangle + |0\rangle|-\frac{1}{2}\rangle) \\ &= \Psi_{-\frac{1}{2}} \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \end{aligned}$$

$$J_- \Psi_{-\frac{1}{2}} = (J_{1-} + J_{2-})(|1\rangle|\frac{1}{2}\rangle, |0\rangle|-\frac{1}{2}\rangle) = (|1\rangle|-\frac{1}{2}\rangle, \sqrt{2}|1\rangle|-\frac{1}{2}\rangle) = \Psi_{-\frac{3}{2}}(1, \sqrt{2})$$

$$\Psi_{-\frac{3}{2}} = (|1\rangle|-\frac{1}{2}\rangle)$$

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$$|\frac{3}{2} \frac{3}{2}\rangle = \Psi_{\frac{3}{2}}(1)$$

$$|\frac{3}{2} \frac{1}{2}\rangle = \frac{1}{\sqrt{3}} J - \Psi_{\frac{3}{2}}(1) = \Psi_{\frac{1}{2}} \begin{pmatrix} \frac{\sqrt{3}}{6} \\ \frac{\sqrt{6}}{3} \end{pmatrix}$$

$$|\frac{3}{2} -\frac{1}{2}\rangle = \frac{1}{2} J - |\frac{3}{2} \frac{1}{2}\rangle = \frac{1}{2} J - \Psi_{\frac{1}{2}} \begin{pmatrix} \frac{\sqrt{3}}{6} \\ \frac{\sqrt{6}}{3} \end{pmatrix} = \Psi_{-\frac{1}{2}} \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{6} \\ \frac{\sqrt{6}}{3} \end{pmatrix} = \Psi_{-\frac{1}{2}} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix}$$

$$|\frac{3}{2} -\frac{3}{2}\rangle = \frac{1}{\sqrt{3}} J - |\frac{3}{2} -\frac{1}{2}\rangle = \frac{1}{\sqrt{3}} J - \Psi_{-\frac{1}{2}} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix} = \Psi_{-\frac{3}{2}}(1, \sqrt{2}) \begin{pmatrix} \frac{1}{3} \\ \frac{\sqrt{2}}{3} \end{pmatrix} = \Psi_{-\frac{3}{2}}(1)$$

表2.3: $\langle j_1 m_1 j_2 m_2 | j m \rangle$, $j_1=1, j_2=\frac{1}{2}, j=\frac{3}{2}$

j	m	m_1	m_2	$\langle j_1 m_1 j_2 m_2 j m \rangle$
$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	1
$\frac{3}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{6}}{3}$
$\frac{3}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{3}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{\sqrt{6}}{3}$
$\frac{3}{2}$	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	1

$|\frac{1}{2} \frac{1}{2}\rangle \perp |\frac{3}{2} \frac{1}{2}\rangle$

次に $j=\frac{1}{2}$ の固有状態を構成する。 $m=m_1+m_2=\frac{1}{2}$ の空間は $\Psi_{\frac{1}{2}}$ で2次元の基底で張られているが、 $\Psi_{\frac{3}{2}}$ から $J-$ を作用させて、 $|\frac{3}{2} \frac{1}{2}\rangle$ として、1状態既に使われているので、 $|\frac{1}{2} \frac{1}{2}\rangle$ をこれと直交する形に定める。

$$|\frac{3}{2} \frac{1}{2}\rangle = \Psi_{\frac{1}{2}} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix}$$

$$|\frac{1}{2} \frac{1}{2}\rangle = \Psi_{\frac{1}{2}} \begin{pmatrix} a \\ b \end{pmatrix}, a^2 + b^2 = 1$$

$$\langle \frac{3}{2} \frac{1}{2} | \frac{1}{2} \frac{1}{2} \rangle = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3} \right) \Psi_{\frac{1}{2}}^\dagger \Psi_{\frac{1}{2}} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\sqrt{3}}{3} a + \frac{\sqrt{6}}{3} b = 0$$

$$a = -b\sqrt{2}, a^2 + b^2 = 3b^2 = 1, a > 0 \text{ とする} \Rightarrow a = \frac{\sqrt{2}}{\sqrt{3}}, b = -\frac{1}{\sqrt{3}}$$

$$|\frac{1}{2} \frac{1}{2}\rangle = \Psi_{\frac{1}{2}} \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} \star$$

$$|\frac{1}{2} -\frac{1}{2}\rangle = J - |\frac{1}{2} \frac{1}{2}\rangle = \Psi_{-\frac{1}{2}} \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix} = \Psi_{-\frac{1}{2}} \begin{pmatrix} -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix} \star$$

表2.4: $\langle j_1 m_1 j_2 m_2 | j m \rangle$, $j_1=1, j_2=\frac{1}{2}, j=\frac{1}{2}$

j	m	m_1	m_2	$\langle j_1 m_1 j_2 m_2 j m \rangle$
$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{\sqrt{6}}{3}$
$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{\sqrt{3}}{3}$
$\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$-\frac{\sqrt{6}}{3}$
$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{\sqrt{3}}{3}$

射影演算子の方法

モジュール見直しより行方別法

2次元の $\Psi_{\frac{1}{2}}$ で張られる空間で規格化されている $|u\rangle = |\frac{3}{2} \frac{1}{2}\rangle = \Psi_{\frac{1}{2}} u$, $u = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix}$ と直交する状態 $|u_{\perp}\rangle$ は u 方向への射影 $P = |u\rangle\langle u|$ を用いて、任意の(一般の)状態 $|t\rangle$ から次式で構成できる。

$$|u_{\perp}\rangle = P'|t\rangle, \quad P' = 1 - P, \quad P = |u\rangle\langle u| \quad P^2 = P, \quad (P')^2 = P'$$

$$\therefore \langle u|u_{\perp}\rangle = \langle u|(1-P)|t\rangle = \langle u|t\rangle - \langle u|u\rangle\langle u|t\rangle = 0$$

ここで $|t\rangle = \Psi_{\frac{1}{2}} t$, $t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ とすれば、 $\langle u|t\rangle = u^{\dagger} \Psi_{\frac{1}{2}}^{\dagger} \Psi_{\frac{1}{2}} t = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\sqrt{3}}{3}$

$$|u_{\perp}\rangle = |t\rangle - |u\rangle\langle u|t\rangle = \Psi_{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \Psi_{\frac{1}{2}} \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} \end{pmatrix} \frac{\sqrt{3}}{3} = \Psi_{\frac{1}{2}} \begin{pmatrix} \frac{2}{3} \\ -\frac{\sqrt{2}}{3} \end{pmatrix}$$

$\langle u_{\perp}|u_{\perp}\rangle = \frac{6}{9} = \frac{2}{3}$ だから、規格化して

$$|\frac{1}{2} \frac{1}{2}\rangle = |u_{\perp}\rangle \frac{1}{\sqrt{\langle u_{\perp}|u_{\perp}\rangle}} = \Psi_{\frac{1}{2}} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}$$

と以前の計算に一致する。

例3: 具体的なクレブッシュゴルダン係数

$$1 \otimes 1 = 2 \oplus 1 \oplus 0$$

m -定の基底:

$$\begin{aligned} \Psi_2 &= (|1\rangle|1\rangle) \\ \Psi_1 &= (|1\rangle|0\rangle, |0\rangle|1\rangle) \\ \Psi_0 &= (|1\rangle|0\rangle, |0\rangle|0\rangle, |1\rangle|1\rangle) \\ \Psi_{-1} &= (|1\rangle|0\rangle, |0\rangle|1\rangle) \\ \Psi_{-2} &= (|1\rangle|1\rangle) \end{aligned}$$

基底の変換

$$J_- \Psi_2 = (J_{1-} + J_{2-}) \langle |1\rangle|1\rangle \rangle = (\sqrt{2}|0\rangle|1\rangle + \sqrt{2}|1\rangle|0\rangle) = \Psi_1 \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

$$J_- \Psi_1 = (J_{1-} + J_{2-}) \langle |1\rangle|0\rangle, |0\rangle|1\rangle \rangle = (\sqrt{2}|0\rangle|0\rangle + \sqrt{2}|1\rangle|1\rangle, \sqrt{2}|1\rangle|1\rangle + \sqrt{2}|0\rangle|0\rangle)$$

$$= \Psi_0 \begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix}$$

$$J_- \Psi_0 = (J_{1-} + J_{2-}) \langle |1\rangle|1\rangle, |0\rangle|0\rangle, |1\rangle|1\rangle \rangle$$

$$= (\sqrt{2}|0\rangle|1\rangle, \sqrt{2}|1\rangle|0\rangle + \sqrt{2}|0\rangle|1\rangle, \sqrt{2}|1\rangle|0\rangle)$$

$$= \Psi_{-1} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$$

$$J_- \Psi_{-1} = (J_{1-} + J_{2-}) \langle |1\rangle|0\rangle, |0\rangle|1\rangle \rangle = (\sqrt{2}|1\rangle|1\rangle, \sqrt{2}|1\rangle|1\rangle)$$

$$= \Psi_{-2} (\sqrt{2}, \sqrt{2})$$

$$|22\rangle = \Psi_2(1)$$

$$|21\rangle = \frac{1}{2} J - \Psi_2(1) = \Psi_1 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$|20\rangle = \frac{1}{\sqrt{6}} J - \Psi_1 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \frac{1}{\sqrt{6}} \Psi_0 \begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \Psi_0 \begin{pmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{pmatrix}$$

$$|2-1\rangle = \frac{1}{\sqrt{6}} J - \Psi_0 \begin{pmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{pmatrix} = \Psi_{-1} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix} = \Psi_{-1} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$|2-2\rangle = \frac{1}{2} J - \Psi_{-1} = \frac{1}{2} \Psi_{-2}(\sqrt{2}, \sqrt{2}) \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \Psi_{-2}(1)$$

表 2.5: $\langle j_1 m_1 j_2 m_2 | j m \rangle$, $j_1 = 1, j_2 = 1, j = 2$

j	m	m_1	m_2	$\langle j_1 m_1 j_2 m_2 j m \rangle$
2	2	1	1	1
2	1	0	0	$\frac{\sqrt{2}}{2}$
2	1	0	1	$\frac{\sqrt{2}}{2}$
2	0	1	-1	$\frac{\sqrt{6}}{6}$
2	0	0	0	$\frac{\sqrt{6}}{3}$
2	0	-1	1	$\frac{\sqrt{6}}{6}$
2	-1	-1	0	$\frac{\sqrt{2}}{2}$
2	-1	0	-1	$\frac{\sqrt{2}}{2}$
2	-2	-1	-1	1

次に、 $j=1$ の固有状態を構成する。 Ψ_1 は2次元の基底であるが、 $|v\rangle = |21\rangle = \Psi_1 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ として1つ使われているので、 $|11\rangle$ をそれと直交する様に定める。

$P = |v\rangle\langle v|$ として射影の方法を用いる。 $|t\rangle = \Psi_1 t$, $t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ とすれば、

$$|v_\perp\rangle = (1-P)|t\rangle = |t\rangle - |v\rangle\langle v|t\rangle = \Psi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \Psi_1 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \frac{\sqrt{2}}{2} = \Psi_1 \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

規格化して

$$|11\rangle = |v_\perp\rangle \sqrt{2} = \Psi_1 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$|10\rangle = \frac{1}{\sqrt{2}} J - |11\rangle = \frac{1}{\sqrt{2}} J - \Psi_1 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \Psi_0 \begin{pmatrix} \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = \Psi_0 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$|1-1\rangle = \frac{1}{\sqrt{2}} J - |10\rangle = \frac{1}{\sqrt{2}} J - \Psi_0 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \Psi_{-1} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = \Psi_{-1} \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

表 2.6: $\langle j_1 m_1 j_2 m_2 | j m \rangle$, $j_1 = 1, j_2 = 1, j = 1$

j	m	m_1	m_2	$\langle j_1 m_1 j_2 m_2 j m \rangle$
1	1	1	0	$\frac{\sqrt{2}}{2}$
1	1	0	1	$-\frac{\sqrt{2}}{2}$
1	0	1	-1	$\frac{\sqrt{2}}{2}$
1	0	0	0	0
1	0	-1	1	$-\frac{\sqrt{2}}{2}$
1	-1	-1	0	$-\frac{\sqrt{2}}{2}$
1	-1	0	-1	$\frac{\sqrt{2}}{2}$

残りは Ψ_0 に属する $|00\rangle$ である。 $|00\rangle$ は $|u\rangle = |20\rangle = \Psi_0 \begin{pmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \end{pmatrix}$ と $|u\rangle = |10\rangle = \Psi_0 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$ とは直交
 である。

任意の (例えば) $|t\rangle = \Psi_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ に対して

$$|u_{\perp}, v_{\perp}\rangle = (1-P)|t\rangle \propto |00\rangle$$

$$\begin{aligned} \therefore \langle u|u_{\perp}, v_{\perp}\rangle &= \langle u|t\rangle - \langle u|(P_u + P_v)|t\rangle \\ &= \langle u|t\rangle - \langle u|P_u|t\rangle = 0 \end{aligned}$$

$$\langle v|u_{\perp}, v_{\perp}\rangle = 0$$

$$\langle u|t\rangle = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) \Psi_0^{\dagger} \Psi_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left(\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\sqrt{6}}{6}$$

$$P_u|t\rangle = |u\rangle \langle u|t\rangle = |u\rangle \frac{\sqrt{6}}{6} = \Psi_0 \begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}$$

$$\langle v|t\rangle = \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right) \Psi_0^{\dagger} \Psi_0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \left(\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{\sqrt{2}}{2}$$

$$P_v|t\rangle = |v\rangle \langle v|t\rangle = |v\rangle \frac{\sqrt{2}}{2} = \Psi_0 \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

$$|u_{\perp}, v_{\perp}\rangle = |t\rangle - P_u|t\rangle - P_v|t\rangle = \Psi_0 \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix} \right) = \Psi_0 \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

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$$|00\rangle = |u_{\perp}, v_{\perp}\rangle \sqrt{3} = \Psi_0 \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

表 2.7: $\langle j_1 m_1 j_2 m_2 | j m \rangle$, $j_1 = 1, j_2 = 1, j = 0$

j	m	m_1	m_2	$\langle j_1 m_1 j_2 m_2 j m \rangle$
0	0	1	-1	$\frac{1}{\sqrt{3}}$
0	0	0	0	$-\frac{1}{\sqrt{3}}$
0	0	-1	1	$\frac{1}{\sqrt{3}}$