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Class 7

2.3.1 極座標での角運動量演算子

$$\mathbb{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}$$

$T = *D^{-1}$ つまり $* = TD$

$\tilde{T}$ とする。 $\tilde{*} = \tilde{D}\tilde{T} = D\tilde{T}$

$$(\mathbb{e}_r, \mathbb{e}_\theta, \mathbb{e}_\phi) \equiv \left(\frac{\partial_r \mathbb{r}}{h_r}, \frac{\partial_\theta \mathbb{r}}{h_\theta}, \frac{\partial_\phi \mathbb{r}}{h_\phi}\right) = \begin{pmatrix} \frac{1}{h_r}\frac{\partial x}{\partial r} & \frac{1}{h_\theta}\frac{\partial x}{\partial \theta} & \frac{1}{h_\phi}\frac{\partial x}{\partial \phi} \\ \frac{1}{h_r}\frac{\partial y}{\partial r} & \frac{1}{h_\theta}\frac{\partial y}{\partial \theta} & \frac{1}{h_\phi}\frac{\partial y}{\partial \phi} \\ \frac{1}{h_r}\frac{\partial z}{\partial r} & \frac{1}{h_\theta}\frac{\partial z}{\partial \theta} & \frac{1}{h_\phi}\frac{\partial z}{\partial \phi} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} D^{-1} = \begin{pmatrix} \cos\theta\sin\phi & \cos\phi\cos\theta & -\sin\theta \\ \sin\phi\sin\theta & \sin\phi\cos\theta & \cos\theta \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} = T$$

$$D = \mathrm{diag}(h_r, h_\theta, h_\phi) = \mathrm{diag}(\underset{h_r}{1}, \underset{h_\theta}{r}, \underbrace{r\sin\theta}_{h_\phi})$$

一方

$$\begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \begin{pmatrix} h_r \tilde{e}_r \\ h_\theta \tilde{e}_\theta \\ h_\phi \tilde{e}_\phi \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

$$= \mathrm{diag}(h_r, h_\theta, h_\phi) \begin{pmatrix} \tilde{e}_r \\ \tilde{e}_\theta \\ \tilde{e}_\phi \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \mathrm{diag}(h_r, h_\theta, h_\phi)\tilde{T} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

$$= D\tilde{T} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$$

逆にとって $\tilde{T}^{-1} = T.$

$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = T \underbrace{\begin{pmatrix} \partial_r \\ \frac{1}{r}\partial_\theta \\ \frac{1}{r\sin\theta}\partial_\phi \end{pmatrix}}_{D^{-1}} = \mathbb{e}_r\partial_r + \mathbb{e}_\theta \frac{1}{r}\partial_\theta + \mathbb{e}_\phi \frac{1}{r\sin\theta}\partial_\phi.$$

$\mathbb{r} = r\mathbb{e}_r$

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$$\mathbb{L} = \mathbb{r}\times\mathbb{P} = -i\hbar r e_r \times \underbrace{\nabla}_{e_r\partial_r + e_\theta \frac{1}{r}\partial_\theta + e_\phi \frac{1}{r\sin\theta}\partial_\phi} = -i\hbar\left(e_\phi \partial_\theta - e_\theta \frac{1}{\sin\theta}\partial_\phi\right)$$

$$= \hbar\begin{pmatrix} i\sin\phi \\ -i\cos\phi \\ 0 \end{pmatrix}\partial_\theta + \hbar\begin{pmatrix} i\cos\phi\cot\theta \\ i\sin\phi\cot\theta \\ -i \end{pmatrix}\partial_\phi$$ ← rによらない

$$L_+ = L_x + iL_y = \hbar(i\sin\phi + \cos\phi)\partial_\theta + (i\cos\phi - \sin\phi)\cot\theta\,\partial_\phi\cdot\hbar$$
$$= \hbar \quad e^{i\phi} \quad \partial_\theta + i \quad e^{i\phi} \quad \cot\theta\,\partial_\phi\cdot\hbar$$
$$= \hbar e^{i\phi}(\partial_\theta + i\cot\theta\,\partial_\phi)$$

$$L_- = \hbar(i\sin\phi - \cos\phi)\partial_\theta + (i\cos\phi + \sin\phi)\cot\theta\,\partial_\phi\cdot\hbar$$
$$= \hbar \quad -e^{-i\phi} \quad \partial_\theta + i \quad e^{-i\phi} \quad \cot\theta\,\partial_\phi\cdot\hbar$$
$$= \hbar\, e^{-i\phi}(-\partial_\theta + i\cot\theta\,\partial_\phi)$$

$$L_z = -i\hbar\partial_\theta \qquad (\partial^\dagger = -\partial)$$

○ 一般論に従い、次の関係をみたす関数を求める

$$\mathbb{L}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}\ ;\ L_z Y_{lm} = \hbar m Y_{lm},\ L_+ Y_{ll} = L_- Y_{ll} = 0$$

$Y_{lm}(\Omega = (\theta,\phi)) = \Theta_{lm}(\theta)\Phi_m(\phi)$ と変数分離形におくと $\underbrace{L_z}_{-i\hbar\partial_\phi} Y_{lm} = \hbar m Y_{lm}$ より

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi} \quad \left(\int_0^{2\pi} d\phi\, |\Phi_m(\phi)|^2 = 1\ で規格化\right)$$

関数の一価性より $e^{i2\pi m} = 1$. つまり mは整数 $(e^{im0} = 1 = e^{i2\pi m})$

よって l も0以上の整数.

$-l \leq m \leq l$

ここでθの関数 $f(\theta)$ に対して

$$L_+[e^{im\phi} f(\theta)] = e^{im\phi}\hbar e^{i\phi}\left[\frac{df}{d\theta} - mf\cot\theta\right]$$

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$$L_-[e^{im\phi}f(\theta)] = -e^{im\phi}\hbar e^{i\phi}[\frac{df}{d\theta} + mf\cot\theta]$$

これは以下のように書き直される

$$L_+[e^{im\phi}f(\theta)] = -\hbar e^{i(m+1)\phi}\sin^{m+1}\theta \frac{d}{d\cos\theta}[\sin^{-m}\theta f(\theta)]$$

$$L_-[e^{im\phi}f(\theta)] = \hbar e^{i(m-1)\phi}\sin^{-(m-1)}\theta \frac{d}{d\cos\theta}[\sin^{m}\theta f(\theta)]$$

これをk回くり返して

$$L_+^k[\,\text{〃}\,] = (-\hbar)^k e^{i(m+k)\phi}\sin^{m+k}\theta \left[\frac{d}{d\cos\theta}\right]^k[\sin^{-m}\theta f(\theta)]$$

$$L_-^k[\,\text{〃}\,] = \hbar^k e^{i(m-k)\phi}\sin^{-(m-k)}\theta \left[\frac{d}{d\cos\theta}\right]^k[\sin^{m}\theta f(\theta)]$$