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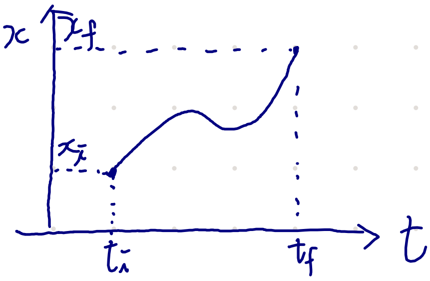
古典系の対称性と保存量の関係：ネーターの定理

解析力学の復習

1D保存力中の質点

$\boxed{F = ma}$, $a = \ddot{x} = \frac{\partial^2 x(t)}{\partial t}$, $F = -\partial_x V$

$t \in [t_i, t_f]$



$x = x(t)$: グラフ
世界線

"運動を定める"

①初期値問題

$x(t_i) = x_0$, $\dot{x}(t_i) = v_0$, x_0, v_0 : given

$\rightarrow x = x(t)$, t_i : 現在→未来を含め定める

②最小作用の原理

$S[x(t)] = \int_{t_i}^{t_f} dt\, L(x(t), \dot{x}(t))$

Action 作用(汎関数) $\delta S = 0$ となる x が実現

functional : 関数→数

c.f. function : 数→数

$\delta S = 0 \Leftrightarrow \frac{\delta L}{\delta x} = \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$ (Euler-Lagrange eq.)

$\because$

$\delta S = \int_{t_i}^{t_f} dt\, \delta L$ ($\delta x(t_i) = \delta x(t_f) = 0$: 端点は固定)

$= \int_{t_i}^{t_f} dt \left[\delta x \frac{\partial L}{\partial x} + \delta \dot{x} \frac{\partial L}{\partial \dot{x}}\right]$

$= \int_{t_i}^{t_f} dt \left[\delta x \frac{\partial L}{\partial x} + \frac{d}{dt}\left(\delta x \frac{\partial L}{\partial \dot{x}}\right) - \delta x \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]$

$= \underbrace{\delta x \frac{\partial L}{\partial \dot{x}}\Big|_{t_i}^{t_f}}_{=0\ (\because\ \partial x(t_i) = \partial x(t_f) = 0)} + \int_{t_i}^{t_f} dt\, \delta x \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]$

$= \int_{t_i}^{t_f} dt\, \delta x \frac{\delta L}{\delta x}$

δx は任意だから $\delta S = 0 \Leftrightarrow \frac{\delta L}{\delta x} = 0$

$F = -\partial_x V$ のとき, $L = \frac{1}{2} m \dot{x}^2 - V$ とする

$\because \frac{\partial L}{\partial x} = -\partial_x V$, $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ $\frac{\delta L}{\delta x} = -\partial_x V - \frac{d}{dt}(m\dot{x})$

$\Leftrightarrow F = m\ddot{x}$ Newton eq.

ハミルトン形式へ

$L(x,\dot{x})$ 独立変数$(x,\dot{x})$

↓

$H(x,p)$ 〃 (x,p)へ変更 $p=\frac{\partial L}{\partial \dot{x}}$

Legendre 変換

$$H=\dot{x}p-L$$

$$\delta H=\delta\dot{x}\,p+\dot{x}\,\delta p-\delta L=\delta\dot{x}\,p+\dot{x}\,\delta p-\frac{\partial L}{\partial x}\delta x-\frac{\partial L}{\partial \dot{x}}\delta\dot{x}$$

$$=\delta\dot{x}\left(p-\frac{\partial L}{\partial \dot{x}}\right)+\dot{x}\,\delta p-\frac{\partial L}{\partial x}\delta x=\dot{x}\,\delta p-\frac{\partial L}{\partial x}\delta x$$

Hは p,x の関数

よって正準方程式

$$\begin{cases}\frac{\partial H}{\partial p}=\dot{x}\\ \frac{\partial H}{\partial x}=-\frac{\partial L}{\partial x}=-\dot{p}\end{cases}$$

ネーターの定理

対称操作

空間並進 ⟺ 運動量保存

時間推進 ⟺ エネルギー保存

空間回転 ⟺ 角運動量保存

。空間推進

$x\to x'=x+\delta a$ とLagrangianを変換

$$L'(x',\dot{x}')\equiv L(x,\dot{x})$$

$$S=\int_{t_i}^{t_f}dt\,L(x,\dot{x})=\int_{t_i}^{t_f}dt\,L'(x',\dot{x}')$$

$$0=\int_{t_i}^{t_f}dt\left(L'(x',\dot{x}')-L(x,\dot{x})\right)$$ ：関数形が不変

$$=\int_{t_i}^{t_f}dt\left(L(x',\dot{x}')-L(x,\dot{x})\right)=\int_{t_i}^{t_f}dt\,\delta L$$

$$=\int_{t_i}^{t_f}dt\left\{\delta x\frac{\partial L}{\partial x}+\delta\dot{x}\frac{\partial L}{\partial \dot{x}}\right\}=\int_{t_i}^{t_f}dt\left\{\delta x\frac{\partial L}{\partial x}+\frac{d}{dt}\left(\delta x\frac{\partial L}{\partial \dot{x}}\right)-\delta x\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right\}$$

$$=\int_{t_i}^{t_f}dt\left\{\frac{d}{dt}\underbrace{\left(\delta x\frac{\partial L}{\partial \dot{x}}\right)}_{=G}+\delta x\,\frac{\delta L}{\delta x}\right\}=\int_{t_i}^{t_f}dt\,\frac{d}{dt}G=G\Big|_{t_i}^{t_f}=0$$

$G=\delta x\frac{\partial L}{\partial \dot{x}}=\delta a\frac{\partial L}{\partial \dot{x}}$ が保存量

$\frac{\partial L}{\partial \dot{x}}=p$ ：運動量が保存する

空間並進対称性の帰結

・空間回転

$\mathbb{r} \to \mathbb{r}' = R\mathbb{r}$, $\tilde{R}R = E_3$ 回転操作

無限小回転 : $R = E_3 + \delta R$ として $\delta\tilde{R} = -\delta R$

$(\delta R)_{ij} = -\varepsilon_{ijk}\delta\omega_k$

$L'(\mathbb{r}', \dot{\mathbb{r}}') \equiv L(\mathbb{r}, \dot{\mathbb{r}})$

L の回転不変性とは $L'(\mathbb{r}', \dot{\mathbb{r}}') = L(\mathbb{r}', \dot{\mathbb{r}}')$

or $L'(\mathbb{r}, \dot{\mathbb{r}}) = L(\mathbb{r}, \dot{\mathbb{r}})$

回転不変な Lagrangian の例

$$L(\mathbb{r}, \dot{\mathbb{r}}) = \frac{1}{2}m\dot{\mathbb{r}}^2 - V(r) \quad , \quad r = |\mathbb{r}| = \sqrt{\tilde{\mathbb{r}}\mathbb{r}}$$

$$= \frac{1}{2}m\tilde{\dot{\mathbb{r}}}\dot{\mathbb{r}} - V(r)$$

$$\because L'(\mathbb{r}, \dot{\mathbb{r}}) \equiv L(\tilde{R}\mathbb{r}, \tilde{R}\dot{\mathbb{r}})$$

$$= \frac{1}{2}m\tilde{\dot{\mathbb{r}}}R\tilde{R}\dot{\mathbb{r}} - V(r) = \frac{1}{2}m\tilde{\dot{\mathbb{r}}}\dot{\mathbb{r}} - V(r)$$

$$= L(\mathbb{r}, \dot{\mathbb{r}})$$

$S = \int_{t_i}^{t_f} dt\, L'(\mathbb{r}', \dot{\mathbb{r}}') = \int_{t_i}^{t_f} L(\mathbb{r}, \dot{\mathbb{r}})$ より

$$0 = \int_{t_i}^{t_f} dt \{L'(\mathbb{r}', \dot{\mathbb{r}}') - L(\mathbb{r}, \dot{\mathbb{r}})\} = \int_{t_i}^{t_f} dt \{L(\mathbb{r}', \dot{\mathbb{r}}') - L(\mathbb{r}, \dot{\mathbb{r}})\}$$

$$= \int_{t_i}^{t_f} dt \{L(\mathbb{r} + \delta\mathbb{r}, \dot{\mathbb{r}} + \delta\dot{\mathbb{r}}) - L(\mathbb{r}, \dot{\mathbb{r}})\} = \int_{t_i}^{t_f} dt\, \delta L$$

$$= \int_{t_i}^{t_f} dt \{\delta r_i \frac{\partial L}{\partial r_i} + \underbrace{\delta\dot{r}_i \frac{\partial L}{\partial \dot{r}_i}}_{\frac{d}{dt}(\delta r_i \frac{\partial L}{\partial \dot{r}_i}) - \delta r_i \frac{d}{dt}(\frac{\partial L}{\partial \dot{r}_i})}\}$$

$$= \int_{t_i}^{t_f} dt \{\frac{d}{dt}G + \delta r_i \frac{\delta L}{\delta r_i}\} = 0$$

実際におきる運動 $\left(\frac{\delta L}{\delta r_i} = 0\right)$ なら

$0 = \int_{t_i}^{t_f} dt \frac{d}{dt} G = G|_{t_f} - G|_{t_i}$ $\quad \therefore G$ が保存

$G = \delta r_i \frac{\partial L}{\partial \dot{r}_i}$, $p_i = \frac{\partial L}{\partial \dot{r}_i}$

$= \delta\mathbb{r} \cdot \mathbb{p}$

$= (\delta\omega \times \mathbb{r}) \cdot \mathbb{p} = (\mathbb{r} \times \mathbb{p}) \cdot \delta\omega = \mathbb{L} \cdot \delta\omega$ が保存

$\delta\omega$ は任意 $\quad \mathbb{L} = \mathbb{r} \times \mathbb{p}$ が保存

・時間推進不変性
一般に変換 $t \to t'$

$q(t) \to q'(t')$ と変換するとき
$\dot{q}(t) \to \dot{q}'(t')$ q : 一般化座標

$$L'(q', \dot{q}', t')dt' = L(q, \dot{q}, t)dt$$

$$S = \int_{t_i}^{t_f} dt\, L(q(t), \dot{q}(t), t) = \int_{t_i}^{t_f} dt' L'(q'(t'), \dot{q}'(t'), t')$$

Lの不変性

$L'(q', \dot{q}', t') = L(q', \dot{q}', t')$ 関数形が不変とすると

$$\int_{t_i}^{t_f} dt\, L(q, \dot{q}, t) = \int_{t_i}^{t_f} dt\, L(q', \dot{q}', t)$$

$$\delta t_f L|_{t_f} - \delta t_i L|_{t_i} + \int_{t_i}^{t_f} dt\, \delta L = 0$$

$$0 = \int_{t_i}^{t_f} dt \left[\frac{d}{dt}\left\{\delta t\, L + \delta q \frac{\partial L}{\partial \dot{q}}\right\} + \delta q \frac{\delta L}{\delta q}\right]$$

$= \int_{t_i}^{t_f} dt \frac{d}{dt} G$ (∵ Newton eq.)

$= G|_{t_f} - G|_{t_i}$ G が保存

時間推進

$$t \to t' = t + \delta t$$

$q'(t') = q(t)$ 関数形は変えない

$$\shortparallel$$

$$q'(t + \delta t) = q'(t) + \delta t\, \dot{q}$$

よって

$$\delta q = q'(t) - q(t) = -\delta t\, \dot{q}$$

$$G = \delta t\, L + (-\delta t\, \dot{q})\frac{\partial L}{\partial \dot{q}} = \delta t\left\{L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right\} = \delta t\{\underbrace{L - \dot{q}p}_{=-H}\}$$

$= -\delta t\, H$ が保存

エネルギー保存則

まとめ

Lagrangianの不変性 ⟶ 保存則
空間推進 → 運動量保存
空間回転 → 角運動量保存
時間推進 → エネルギー保存
⟹ 量子論と整合的