5/12 第4回 201710848 有富洸人

解析力学

- motion eq. $F = m\ddot{x}$
- 保存力 $F = -\partial_x V$

1、初期値問題

x_0, v_0

2、最小作用の原理

$$S[x(t)] \equiv \int_{t_i}^{t_f} dt\, L(x(t), \dot{x}(t))$$

$\rightarrow \delta S = 0$ となる

$$\Rightarrow \frac{\delta L}{\delta x} \equiv \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \quad \text{(Euler-Lagrange eq.)}$$

$F = -\partial_x V$ のとき $L = \frac{1}{2}m\dot{x}^2 - V$ とする。

$$\rightarrow \frac{\delta L}{\delta x} = -\partial_x V - \frac{d}{dt} m\dot{x} = \underbrace{F - m\ddot{x} = 0}_{\text{Newton}}$$

正準方程式へ

$$L(x,\dot{x}) \to H(x,p) \qquad \left(p \equiv \frac{\partial L}{\partial \dot{x}}\right)$$

Legendre変換 $H \equiv \dot{x}p - L$

$$\begin{aligned}
\to \delta H &= \delta\dot{x}\,p + \dot{x}\,\delta p - \delta L \\
&= \delta\dot{x}\,p + \dot{x}\,\delta p - \frac{\partial L}{\partial x}\delta x - \frac{\partial L}{\partial \dot{x}}\delta\dot{x} \\
&= \delta\dot{x}\left(p - \frac{\partial L}{\partial \dot{x}}\right) + \dot{x}\,\delta p - \frac{\partial L}{\partial x}\delta x \\
&= \dot{x}\,\delta p - \frac{\partial L}{\partial x}\delta x
\end{aligned}$$

$\to$ Hはδp, δxの関数

- 正準方程式 $\dfrac{\partial H}{\partial p} = \dot{x}$, $\dfrac{\partial H}{\partial x} = -\dfrac{\partial L}{\partial x} = -\dot{p}$

$$\begin{aligned}
\to H = \dot{x}p - L &= m\dot{x}^2 - \left(\frac{1}{2}m\dot{x}^2 - V\right) \\
&= \frac{p^2}{2m} + V
\end{aligned}$$

$$\to \frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}, \quad \frac{\partial H}{\partial x} = \partial_x V = -\dot{p}$$

$$\therefore \underbrace{m\ddot{x} = -\partial_x V = F}_{\text{Newton eq.}}$$

ネーターの定理

・空間並進

$$x \mapsto x' = x + \delta a$$ と変換

$$L'(x', \dot{x}') = L(x, \dot{x})$$

$$S = \int_{t_i}^{t_f} dt\, L(x, \dot{x}) = \int_{t_i}^{t_f} dt\, L'(x', \dot{x}')$$

$$0 = \int_{t_i}^{t_f} dt \left\{ L'(x', \dot{x}') - L(x, \dot{x}) \right\}$$

↓ L 不変

$$= \int_{t_i}^{t_f} dt \left\{ \underbrace{L(x', \dot{x}') - L(x, \dot{x})}_{\delta L} \right\}$$

$$= \int_{t_i}^{t_f} dt\, \delta L$$

$$= \int_{t_i}^{t_f} dt \left\{ \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right\}$$

$$= \int_{t_i}^{t_f} dt \left\{ \frac{\partial L}{\partial x} \delta x + \frac{d}{dt}\left(\delta x \frac{\partial L}{\partial \dot{x}} \right) - \delta x \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \right\}$$

$$= \int_{t_i}^{t_f} dt \left\{ \frac{d}{dt}\left(\underbrace{\delta x \frac{\partial L}{\partial \dot{x}}}_{G} \right) + \overset{0}{\frac{\delta L}{\delta x}} \delta x \right\}$$

$$= \int_{t_i}^{t_f} dt \cdot \frac{d}{dt} G \qquad = G \Big|_{t_i}^{t_f} = 0$$

$$G = \delta x \frac{\partial L}{\partial \dot{x}} = \delta a \frac{\partial L}{\partial \dot{x}}$$ が保存量

$\rightarrow \frac{\partial L}{\partial \dot{x}} = p$: 運動量

→ 空間並進では運動量が保存する。

∘ 空間回転

$$\mathbb{r} \mapsto \mathbb{r}' = R\mathbb{r}, \quad \tilde{R}R = E_3$$

$$L'(\mathbb{r}', \dot{\mathbb{r}}') = L(\mathbb{r}, \dot{\mathbb{r}})$$

L の回転不変性 : $\begin{cases} L'(\mathbb{r}', \dot{\mathbb{r}}') = L(\mathbb{r}', \dot{\mathbb{r}}') \\ L'(\mathbb{r}, \dot{\mathbb{r}}) = L(\mathbb{r}, \dot{\mathbb{r}}) \end{cases}$

$$S = \int_{t_i}^{t_f} dt\, L'(\mathbb{r}', \dot{\mathbb{r}}') = \int_{t_i}^{t_f} dt\, L(\mathbb{r}, \dot{\mathbb{r}})$$

$$\begin{aligned} 0 &= \int_{t_i}^{t_f} dt \left\{ L'(\mathbb{r}', \dot{\mathbb{r}}') - L(\mathbb{r}, \dot{\mathbb{r}}) \right\} \\ &= \int_{t_i}^{t_f} dt \left\{ L(\mathbb{r}', \dot{\mathbb{r}}') - L(\mathbb{r}, \dot{\mathbb{r}}) \right\} \\ &\quad \text{↓ 無限小変換} \\ &= \int_{t_i}^{t_f} dt \left\{ L(\mathbb{r} + \delta\mathbb{r}, \dot{\mathbb{r}} + \delta\dot{\mathbb{r}}) - L(\mathbb{r}, \dot{\mathbb{r}}) \right\} \\ &= \int_{t_i}^{t_f} dt\, \delta L \end{aligned}$$

$$= \int_{t_i}^{t_f} dt \left\{ \delta r_i \frac{\partial L}{\partial r_i} + \delta \dot{r}_i \frac{\partial L}{\partial \dot{r}_i} \right\}$$

$$= \int_{t_i}^{t_f} dt \left\{ \frac{d}{dt} G + \delta r_i \frac{\delta L}{\delta r_i} \right\} = 0$$

Euler-Lagrange eq.

$$= \int_{t_i}^{t_f} dt \frac{d}{dt} G$$

$$= G|_{t_f} - G|_{t_i} = 0$$

→ G が保存

$$G = \delta r_i \frac{\partial L}{\partial \dot{r}_i} \quad , \quad p_i = \frac{\partial L}{\partial \dot{r}_i}$$

$$= \delta \mathbb{r} \cdot \mathbb{p}$$

$$= (\delta \omega \times \mathbb{r}) \cdot \mathbb{p}$$

$$= (\mathbb{r} \times \mathbb{p}) \cdot \delta \omega$$

$= \mathbb{L} \cdot \delta \omega$ が保存.

→ $\mathbb{L} = \mathbb{r} \times \mathbb{p}$: 角運動量が保存.

○ 時間推進

$$t \to t'$$
$$q(t) \to q'(t')$$
$$\dot{q}(t) \to \dot{q}'(t')$$

$$\to \quad L'(q', \dot{q}', t')\,dt' = L(q, \dot{q}, t)\,dt$$

$$S = \int_{t_i}^{t_f} dt\, L(q(t), \dot{q}(t), t)$$
$$= \int_{t_i'}^{t_f'} dt'\, L'(q'(t'), \dot{q}'(t'), t')$$
$$= \int_{t_i'}^{t_f'} dt\, L'(q'(t), \dot{q}'(t), t)$$

Lの不変性 : $L'(q', \dot{q}', t) = L(q', \dot{q}', t')$

$$\to \int_{t_i}^{t_f} dt\, L(q, \dot{q}, t) = \int_{t_i'}^{t_f'} dt\, L(q', \dot{q}', t)$$

$$\delta t_f L\Big|_{t_f} - \delta t_i L\Big|_{t_i} + \int_{t_i}^{t_f} dt\, \delta L = 0$$

$$0 = \int_{t_i}^{t_f} dt \left[\frac{d}{dt}\left\{ \delta t L + \delta q \frac{\partial L}{\partial q} \right\} + \delta q \frac{\delta L}{\delta q} \right]$$

Newton eq.

$$= \int_{t_i}^{t_f} dt\, \frac{d}{dt} G$$

$$= G\Big|_{t_f} - G\Big|_{t_i}$$

→ G が保存. $G = \delta t L + \delta q \frac{\partial L}{\partial \dot{q}}$

$$\rightarrow G = \delta t L + (-\delta t \dot{q}) \frac{\partial L}{\partial \dot{q}}$$

$$= \delta t \left\{ L - \dot{q} \frac{\partial L}{\partial \dot{q}} \right\}$$

$$= \delta t \left\{ L - \dot{q} p \right\}$$

$$= -\delta t H$$ が保存.

→ エネルギー保存則.

◦ Lagrangian の不変性 → 保存則

⇒ 量子論と整合的.