力学　物理 201110833　青柳 航

$$(\mathbb{A}\times\mathbb{B})\cdot\mathbb{A} = (\varepsilon_{ijk}A_jB_k)\cdot A_i = \varepsilon_{ijk}A_iA_jB_k$$

$$= \frac{1}{2}\varepsilon_{ijk}(A_iA_j + A_jA_i)B_k$$

$$= \frac{1}{2}\underbrace{(\varepsilon_{ijk} + \varepsilon_{jik})}_{=0}A_iA_jB_k$$

$$= 0$$

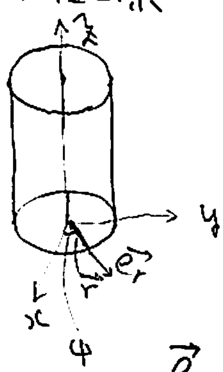
$$|\vec{A}\times\vec{B}| = |\vec{A}|\cdot|\vec{B}|\sin\theta$$

$$\vec{A}\parallel\vec{B} \to \theta = 0, \pi \Rightarrow \vec{A}\times\vec{B} = 0$$

$\vec{e}_1\times\vec{e}_2 = \vec{e}_3$ などを考えると $\vec{A}\times\vec{B}$：$\vec{A}\to\vec{B}$ へ右ねじをまわすとき ねじの進行方向

座標変換の例

円柱座標　(2次の極座標)



$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$z = z$$

$$\vec{e}_r = \widehat{\frac{\partial\vec{r}}{\partial r}} = \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|}$$

$$\vec{r} = \vec{e}_1 x + \vec{e}_2 y + \vec{e}_3 z$$

$$= (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

$$\frac{\partial \vec{r}}{\partial r} = (\vec{e_1}, \vec{e_2}, \vec{e_3})\begin{pmatrix}\cos\varphi \\ \sin\varphi \\ 0\end{pmatrix}$$

$$\left|\frac{\partial \vec{r}}{\partial r}\right| = \cos^2\varphi + \sin^2\varphi + 0 = 1$$

$$\vec{e}_r = (\vec{e_1}, \vec{e_2}, \vec{e_3})\, \mathbb{e}_r$$

$$= \mathcal{O}\, \mathbb{e}_r \qquad \mathbb{e}_r = \begin{pmatrix}\cos\varphi \\ \sin\varphi \\ 0\end{pmatrix}$$

$$\vec{e_\varphi} = \widehat{\frac{\partial \vec{r}}{\partial \varphi}} = \frac{\frac{\partial \vec{r}}{\partial \varphi}}{\left|\frac{\partial \vec{r}}{\partial \varphi}\right|}$$

$$\frac{\partial \vec{r}}{\partial \varphi} = (\vec{e_1}, \vec{e_2}, \vec{e_3})\begin{pmatrix}-r\sin\varphi \\ r\cos\varphi \\ 0\end{pmatrix}$$

$$\left|\frac{\partial \vec{r}}{\partial \varphi}\right|^2 = \left|\frac{\partial r}{\partial \varphi}\right|^2 = r^2\sin^2\varphi + r^2\cos^2\varphi + 0$$

$$= r^2 \qquad \left|\frac{\partial \vec{r}}{\partial \varphi}\right| = r$$

$$\vec{e_\varphi} = (\vec{e_1}, \vec{e_2}, \vec{e_3})\begin{pmatrix}-\sin\varphi \\ \cos\varphi \\ 0\end{pmatrix}, \quad \mathbb{e}_\varphi = \begin{pmatrix}-\sin\varphi \\ \cos\varphi \\ 0\end{pmatrix}$$

$$= \mathcal{O}\, \mathbb{e}_\varphi$$

$$\vec{e}_z = \vec{e_3} = (\vec{e_1}, \vec{e_2}, \vec{e_3})\begin{pmatrix}0 \\ 0 \\ 1\end{pmatrix} = \mathcal{O}\, \mathbb{e}_z \qquad \mathbb{e}_z = \begin{pmatrix}0 \\ 0 \\ 1\end{pmatrix}$$

$$(\vec{e_r}\ \vec{e_\varphi}\ \vec{e_z}) = (\vec{e_1}\ \vec{e_2}\ \vec{e_3}) = \begin{pmatrix}\mathbb{e}_r & \mathbb{e}_\varphi & \mathbb{e}_z\end{pmatrix}$$

$$= \mathcal{O}\, T_{r\varphi z},$$

$$T_{r\varphi z} = \begin{pmatrix}\cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1\end{pmatrix}$$

一般にベクトル$\vec{v}$

$$\vec{v} = \vec{e}_1 v_x + \vec{e}_2 v_y + \vec{e}_3 v_z$$

$$= (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \mathcal{O}\mathbb{v}$$

$$= (\vec{e}_r, \vec{e}_\varphi, \vec{e}_z)\begin{pmatrix} v_r \\ v_\varphi \\ v_z \end{pmatrix} = \vec{e}_r v_r + \vec{e}_\varphi v_\varphi + \vec{e}_z v_z$$

$\left(\begin{pmatrix} v_r \\ v_\varphi \\ v_z \end{pmatrix} \text{円柱座標での成分}\right)$ $\quad \mathbb{v} = T_{r\varphi z}\begin{pmatrix} v_r \\ v_\varphi \\ v_z \end{pmatrix}$

$\begin{pmatrix} v_r \\ v_\varphi \\ v_z \end{pmatrix} = \tilde{T}\mathbb{v}$

$\vec{r}$に質点が存在している時の速度ベクトル $\vec{V}$

加速度ベクトル $\vec{a}$

は円柱座標系$\mathcal{O}_{r\varphi z} = (\vec{e}_r, \vec{e}_\varphi, \vec{e}_z)$でどう表されるか？

以上は一般のベクトル

$$\vec{r} = \vec{e}_1 x + \vec{e}_2 y + \vec{e}_3 z = \mathcal{O}\mathbb{r}$$

$$\vec{V} = \dot{\vec{r}} = \mathcal{O}\dot{\mathbb{r}} = \mathcal{O}\mathbb{V}$$

$$\mathbb{V} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \qquad x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$z = z$$

$$\dot{x} = \frac{d}{dt}x(t) \qquad \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$$

$$= \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$$

よって 一般のベクトルの変換則より

$$\begin{pmatrix} V_r \\ V_\varphi \\ V_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{r}\cos\varphi - r\dot{\varphi}\sin\theta \\ \dot{r}\sin\varphi + r\dot{\varphi}\cos\theta \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \dot{r} \\ r\dot{\varphi} \\ \dot{z} \end{pmatrix}$$

$$\vec{V} = (\vec{e_r}, \vec{e_\varphi}, \vec{e_z}) \begin{pmatrix} \dot{r} \\ r\dot{\varphi} \\ \dot{z} \end{pmatrix}$$

加速度ベクトル

$$\vec{a} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} \ddot{r}\cos\varphi - \dot{r}\dot{\varphi}\sin\varphi - \dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}\dot{\varphi}\cos\varphi \\ \ddot{r}\sin\varphi + \dot{r}\dot{\varphi}\cos\varphi + \dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\sin\varphi - r\dot{\varphi}\dot{\varphi}\sin\varphi \\ \ddot{z} \end{pmatrix}$$

$$\mathbb{a} = \begin{pmatrix} (\ddot{r} - r\dot{\varphi}^2)\cos\varphi - (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\sin\varphi \\ (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\cos\varphi + (\ddot{r} - r\dot{\varphi}^2)\sin\varphi \\ \ddot{z} \end{pmatrix}$$

**より

$$\begin{pmatrix} a_r \\ a_\varphi \\ a_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{a}$$

$$= \begin{pmatrix} \ddot{r} - r\dot{\varphi}^2 \\ 2\dot{r}\dot{\varphi} + r\dot{\varphi} \\ \dot{z} \end{pmatrix}$$

$$\vec{a} = (\vec{e_r}, \vec{e_\varphi}, \vec{e_z}) \begin{pmatrix} \ddot{r} - \dot{r}\dot{\varphi} \\ 2\dot{r}\dot{\varphi} + r\dot{\varphi}^2 \\ \ddot{z} \end{pmatrix}$$

$\Rightarrow \mathbb{F} = (\vec{e_r}, \vec{e_\varphi}, \vec{e_z}) \begin{pmatrix} F_r \\ F_\varphi \\ F_z \end{pmatrix}$ とすれば

$$F_r = m(\ddot{r} - \dot{r}\dot{\varphi})$$

$$F_\varphi = m(2\dot{r}\dot{\varphi} + r\dot{\varphi}^2)$$

$$F_z = m\ddot{z}$$

円柱座標での運動方程式

$$\vec{0} = \vec{r}\times\vec{P} = \vec{r}\times\frac{d}{dt}\vec{P}$$

$m\frac{d\vec{r}}{dt}$

$$= \frac{d}{dt}\vec{r}\times\vec{P} = \frac{d\vec{r}}{dt}\times\vec{P} = \vec{0}$$

$\vec{F}\propto\vec{r}$ なら（中心力のみが働らく場合）

$\vec{0} = \frac{d\vec{L}}{dt}$　　$\vec{L}=\vec{r}\times\vec{p}$　．角運動量（運動量のモーメント）

↳角運動量は定数ベクトル：時間に依存しない、保存量

運動の定数

$\vec{L}=\vec{c}$（定数ベクトル）

$\vec{F}$を円柱座標で表現する

←中心力

$\vec{F}=F_r\vec{e_r}$　（$\vec{e_\phi}$，$\vec{e_z}$ は現れない）

$F_\phi=0, F_z=0$

成分ごとの方程式

$F_r = ma_r$

$0 = F_\phi = ma_\phi = m(2\dot{r}\dot{\phi}+r\ddot{\phi})$

$0 = F_z = ma_z = m\ddot{z}$

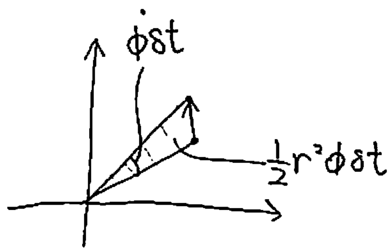
$0 = m\ddot{z} \rightarrow z = v_0 t + z_0$

z方向へは等速運動

$2\dot{r}\dot{\phi}+r\ddot{\phi} = \frac{1}{r}(2r\dot{r}\dot{\phi}+r^2\ddot{\phi})$

$= \frac{1}{r}\frac{d}{dt}r^2\dot{\phi}$

$\phi\delta t\times r$



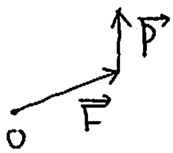
$F_\phi = 0 = m\frac{1}{r}\frac{d}{dt}r^2\dot{\phi}$

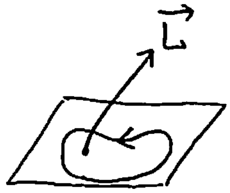
$\frac{d}{dt}r^2\dot{\phi}=0$，$r^2\dot{\phi}=$定数

単位時間あたり動径が通る部分の面積一定

面積速度一定の法則

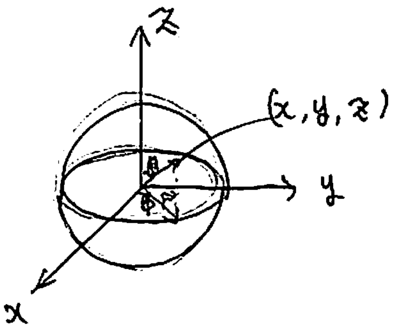
中心力 → $L = \vec{r}\times\vec{p}$ 一定





$L \perp \vec{r}$
$L \perp \vec{r}$ → 質点は $\vec{L}$ を法線とする平面内の運動を行う

☆極座標(3D)



$x = r\sin\theta\cos\phi$　　$\phi : 0\to 2\pi$

$y = r\sin\theta\sin\phi$　　$\theta : 0 \to \pi$

$z = r\cos\theta$

$$\vec{e_r} = \widehat{\frac{\partial\vec{r}}{\partial r}} = \theta\,\widehat{\frac{\partial\mathbf{r}}{\partial r}} = \theta\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix} \quad \frac{\partial\mathbf{r}}{\partial r} = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\phi\end{pmatrix}$$

$$\vec{r} = (\vec{e_1}\ \ \vec{e_2}\ \ \vec{e_3})\,\mathbf{r} = \theta\,\mathbf{r} \quad \mathbf{r} = \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}r\sin\theta\cos\phi\\ r\sin\theta\sin\phi\\ r\cos\theta\end{pmatrix} \quad \left|\frac{\partial\mathbf{r}}{\partial r}\right|^2 = \sin^2\theta$$

$(\sin^2\theta+\cos^2\theta)$
$+\cos^2\theta = 1$

$$\vec{e_\theta} = \widehat{\frac{\partial\vec{r}}{\partial\theta}} = \theta\,\widehat{\frac{\partial\mathbf{r}}{\partial\theta}}$$

$$\frac{\partial\mathbf{r}}{\partial\theta} = \begin{pmatrix}r\cos\theta\cos\phi\\ r\cos\theta\sin\phi\\ -r\sin\theta\end{pmatrix} \quad \left|\frac{\partial\mathbf{r}}{\partial\theta}\right|^2 = r^2$$

$$\vec{e_\theta} = \theta\begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix}$$

$$\vec{e_\phi} = \widehat{\frac{\partial\vec{r}}{\partial\phi}} = \theta\,\widehat{\frac{\partial\mathbf{r}}{\partial\phi}}$$

$$\frac{\partial\mathbf{r}}{\partial\phi} = \begin{pmatrix}-r\sin\theta\sin\phi\\ r\sin\theta\cos\phi\\ 0\end{pmatrix} \quad \left|\frac{\partial\mathbf{r}}{\partial\phi}\right|^2 = r^2\sin^2\theta$$

$$\left|\frac{\partial\mathbf{r}}{\partial\phi}\right| = r\sin\theta$$

$$\vec{e_\phi} = \theta\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}$$

任意のベクトル $\vec{v}$

$$\vec{v} = \mathcal{O}\, v = \mathcal{O}_{r\theta\phi}\, v_{r\theta\phi}$$

$$= \mathcal{O}\, T\, v_{r\theta\phi}$$

$$(\vec{e}_r\ \vec{e}_\theta\ \vec{e}_\phi) = \mathcal{O}\underbrace{\big((e_r)(e_\theta)(e_\phi)\big)}$$

$$T = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\phi & 0 \end{pmatrix}$$

${}^tT\,T = E_3$

直交行列

$$v = T_{r\theta\phi}\, v_{r\theta\phi}$$

質点の速度、加速度

$\vec{v}$ =円柱座標、極座標表示