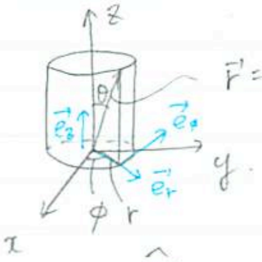
力学A 6/9 レポート

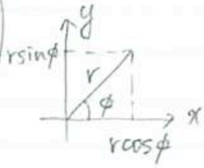
201110853 栗木美香

座標変換の例

・円柱座標（2次の極座標）



$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$



$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases}$$

$$\begin{cases} \vec{e}_r = \widehat{\frac{\partial \vec{r}}{\partial r}} = \dfrac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} \qquad \vec{r} = \vec{e}_1 x + \vec{e}_2 y + \vec{e}_3 z = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} r\cos\phi \\ r\sin\phi \\ z \end{pmatrix} \\ \dfrac{\partial \vec{r}}{\partial r} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \qquad \left|\dfrac{\partial \vec{r}}{\partial r}\right|^2 = \cos^2\phi + \sin^2\phi + 0 = 1 \\ \vec{e}_r = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} = \mathcal{O}\mathbb{e}_r \qquad \mathbb{e}_r = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \end{cases}$$

$$\begin{cases} \vec{e}_\phi = \widehat{\frac{\partial \vec{r}}{\partial r}} = \dfrac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} \qquad \dfrac{\partial \vec{r}}{\partial \phi} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} -r\sin\phi \\ r\cos\phi \\ 0 \end{pmatrix} \\ \left|\dfrac{\partial \vec{r}}{\partial \phi}\right|^2 = \left|\dfrac{\partial \mathbb{r}}{\partial \phi}\right|^2 = r^2\sin^2\phi + r^2\cos^2\phi + 0 = r^2 \qquad \left|\dfrac{\partial \vec{r}}{\partial \phi}\right| = r \\ \vec{e}_\phi = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} = \mathcal{O}\mathbb{e}_\phi \qquad \mathbb{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \end{cases}$$

$$\left\{ \vec{e}_z = \vec{e}_3 = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathcal{O}\mathbb{e}_z \qquad \mathbb{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right.$$

一般にベクトル $\vec{v}$

$$\vec{v} = \vec{e}_1 v_1 + \vec{e}_2 v_2 + \vec{e}_3 v_3 = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathcal{O}\mathbb{v}$$

$$= (\vec{e}_r, \vec{e}_\phi, \vec{e}_z)\underbrace{\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix}}_{\text{円柱座標での成分}} = \vec{e}_r v_r + \vec{e}_\phi v_\phi + \vec{e}_z v_z$$

$$(\vec{e}_r, \vec{e}_\phi, \vec{e}_z) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\underbrace{\begin{pmatrix} \mathbb{e}_1 \end{pmatrix}\begin{pmatrix} \mathbb{e}_2 \end{pmatrix}\begin{pmatrix} \mathbb{e}_3 \end{pmatrix}}_{3\times3\text{行列} = T_{r\phi z}} = \mathcal{O}\,T_{r\phi z}$$

$$T_{r\phi z} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tilde{T}T = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} & T & \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_3 \quad (\tilde{T} = T^{-1})$$

$$\because \vec{v} = \mathcal{O} T_{r\phi z} \begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = \mathcal{O}\mathbb{v}$$

$$\therefore \mathbb{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = T_{r\phi z} \begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix}$$

$$\begin{pmatrix} v_r \\ v_\phi \\ v_z \end{pmatrix} = \tilde{T}_{r\phi z} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \text{—} (**)$$

$\vec{r}$に質点が存在しているときの速度ベクトル$\vec{V}$，加速度ベクトル$\vec{\alpha}$は，

円柱座標系$\mathcal{O}_{r\phi z} = (\vec{e}_r, \vec{e}_\phi, \vec{e}_z)$ でどう表わされるか？

$$\vec{r} = \vec{e}_1 x + \vec{e}_2 y + \vec{e}_3 z = \mathcal{O}\mathbb{r}$$
$$\vec{V} = \dot{\vec{r}} = \mathcal{O}\dot{\mathbb{r}} = \mathcal{O}\mathbb{V} \qquad \mathbb{V} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \quad \begin{cases} x = r\cos\phi \\ y = r\sin\phi \\ z = z \end{cases}$$

$x = x(t) = r(t)\cos\phi(t)$
[rとϕは時間tの関数]

$$\begin{cases} \dot{x} = \frac{d}{dt}x(t) = \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ \dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\ \dot{z} = 0 \end{cases}$$

よって一般ベクトルの変換則(**)より

$$\begin{pmatrix} V_r \\ V_\phi \\ V_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{r}\cos\phi - r\dot{\phi}\sin\phi \\ \dot{r}\sin\phi + r\dot{\phi}\cos\phi \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \dot{r}\cos^2\phi - r\dot{\phi}\sin\phi\cos\phi + \dot{r}\sin^2\phi + r\dot{\phi}\sin\phi\cos\phi \\ r\dot{\phi}\sin^2\phi - \dot{r}\sin\phi\cos\phi + r\dot{\phi}\cos^2\phi + \dot{r}\sin\phi\cos\phi \\ \dot{z} \end{pmatrix}$$

$$= \begin{pmatrix} \dot{r} \\ r\dot{\phi} \\ \dot{z} \end{pmatrix} \qquad \therefore \vec{V} = (\vec{e}_r, \vec{e}_\phi, \vec{e}_z) \begin{pmatrix} \dot{r} \\ r\dot{\phi} \\ \dot{z} \end{pmatrix}$$

加速度ベクトル

$$a=\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix}=\begin{pmatrix}\ddot{r}\cos\phi-\dot{r}\dot{\phi}\sin\phi-\dot{r}\dot{\phi}\sin\phi-r\ddot{\phi}\sin\phi-r\dot{\phi}^2\cos\phi\\ \ddot{r}\sin\phi+\dot{r}\dot{\phi}\cos\phi+\dot{r}\dot{\phi}\cos\phi+r\ddot{\phi}\cos\phi-r\dot{\phi}^2\sin\phi\\ \ddot{z}\end{pmatrix}$$

$\vec{a}=\Theta a$ より

$$\begin{pmatrix}a_r\\ a_\phi\\ a_z\end{pmatrix}=\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix} a=\begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\ddot{r}-r\dot{\phi}^2\\ 2\dot{r}\dot{\phi}+r\ddot{\phi}\\ \ddot{z}\end{pmatrix}$$

$$\vec{a}=(\vec{e}_r,\vec{e}_\phi,\vec{e}_z)\begin{pmatrix}\ddot{r}-r\dot{\phi}^2\\ 2\dot{r}\dot{\phi}+r\ddot{\phi}\\ \ddot{z}\end{pmatrix}$$

$\mathbb{F}=(\vec{e}_r,\vec{e}_\phi,\vec{e}_z)\begin{pmatrix}F_r\\ F_\phi\\ F_z\end{pmatrix}$ とすれば,

$$\begin{cases}F_r=m(\ddot{r}-r\dot{\phi}^2)\\ F_\phi=m(2\dot{r}\dot{\phi}+r\ddot{\phi})\\ F_z=m\ddot{z}\end{cases}\quad\leftarrow\text{[円柱座標での運動方程式]}$$