力学A 演習問題 4 レポート

I. I.1 $\tilde{f}(x,y) = f(X(x,y), Y(x,y))$ より

$$\frac{\delta\tilde{f}}{\delta x} = \frac{\delta f}{\delta x}$$

微小量の関係式を代入 $= \frac{1}{\delta x}\left(\frac{\partial f}{\partial X}\delta X + \frac{\partial f}{\partial Y}\delta Y\right)$

$$= \frac{1}{\delta x}\left\{\frac{\partial f}{\partial X}\left(\frac{\partial X}{\partial x}\delta x + \frac{\partial X}{\partial y}\delta y\right) + \frac{\partial f}{\partial Y}\left(\frac{\partial Y}{\partial x}\delta x + \frac{\partial Y}{\partial y}\delta y\right)\right\}$$

$$= \frac{\partial X}{\partial x}\frac{\partial f}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial f}{\partial Y} + \left(\frac{\partial f}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial y}\right)\frac{\delta y}{\delta x}$$

$\delta x \to 0$ とすると $\frac{\partial\tilde{f}}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial f}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial f}{\partial Y} + \underbrace{\left(\frac{\partial f}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial y}\right)\frac{\partial y}{\partial x}}_{=0}$

$\frac{\partial\tilde{f}}{\partial x} = \frac{\partial f}{\partial x}$ より $= \frac{\partial X}{\partial x}\frac{\partial\tilde{f}}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial\tilde{f}}{\partial Y}$ //

$$\frac{\delta\tilde{f}}{\delta y} = \frac{\delta f}{\delta y}$$

同様にして $= \frac{\partial X}{\partial y}\frac{\partial f}{\partial X} + \frac{\partial Y}{\partial y}\frac{\partial f}{\partial Y} + \left(\frac{\partial f}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial x}\right)\frac{\delta x}{\delta y}$

$\delta y \to 0$ とすると $\frac{\partial\tilde{f}}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial X}{\partial y}\frac{\partial f}{\partial X} + \frac{\partial Y}{\partial y}\frac{\partial f}{\partial Y} + \underbrace{\left(\frac{\partial f}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial x}\right)\frac{\partial x}{\partial y}}_{=0}$

$\frac{\partial\tilde{f}}{\partial y} = \frac{\partial f}{\partial y}$ より $= \frac{\partial X}{\partial y}\frac{\partial\tilde{f}}{\partial X} + \frac{\partial Y}{\partial y}\frac{\partial\tilde{f}}{\partial Y}$ //

I.2 I.1 より $\frac{\partial}{\partial x_i} = \frac{\partial X_1}{\partial x_i}\frac{\partial}{\partial X_1} + \frac{\partial X_2}{\partial x_i}\frac{\partial}{\partial X_2} + \cdots + \frac{\partial X_n}{\partial x_i}\frac{\partial}{\partial X_n}$

Einsteinの記法より

$$\frac{\partial}{\partial x_i} = \frac{\partial X_j}{\partial x_i}\frac{\partial}{\partial X_j}$$ //

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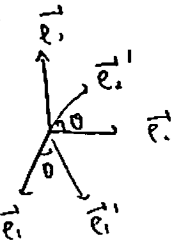
II. II1. II1-1



$$\vec{e}_1{}' = \vec{e}_1\cos\theta + \vec{e}_2\sin\theta$$
$$\vec{e}_2{}' = -\vec{e}_1\sin\theta + \vec{e}_2\cos\theta$$
$$\vec{e}_3{}' = \vec{e}_3$$

II1-2

$$(\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}') = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\therefore\ e_1' = \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix},\quad e_2' = \begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix},\quad e_3' = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

II1-3

$$\begin{cases}\vec{e}_3 = \vec{e}_3{}' \\ \vec{e}_1 = \vec{e}_1{}'\cos\theta - \vec{e}_2{}'\sin\theta \\ \vec{e}_2 = \vec{e}_1{}'\sin\theta + \vec{e}_2{}'\cos\theta\end{cases}$$

$v = Tv'$ より
$$T = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$T\cdot\tilde{T} = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$
$$= \begin{pmatrix}\cos^2\theta + \sin^2\theta & 0 & 0\\ 0 & \sin^2\theta + \cos^2\theta & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$
$$= E_3$$

∴ Tは直交行列である

II-2　Tは直交行列だから　$\tilde{T} = T^{-1}$

$$\therefore\ \mathbb{r} = T\,\mathbb{r}'$$

$$\mathbb{r}' = T^{-1}\mathbb{r} = \tilde{T}\,\mathbb{r}$$

//

II-3.　$\nabla = \left(\dfrac{\partial}{\partial x_i}\right)_i$,　$\nabla' = \left(\dfrac{\partial}{\partial x_i'}\right)_i$　とする

I.2 より　$\dfrac{\partial}{\partial x_i} = \dfrac{\partial}{\partial x_j'}\dfrac{\partial x_j'}{\partial x_i}$

$$= \left(\frac{\partial x_1'}{\partial x_i}\ \ \frac{\partial x_2'}{\partial x_i}\ \cdots\ \frac{\partial x_n'}{\partial x_i}\right)\begin{pmatrix}\frac{\partial}{\partial x_1'}\\ \frac{\partial}{\partial x_2'}\\ \vdots\\ \frac{\partial}{\partial x_n'}\end{pmatrix}$$

$$\therefore\ \nabla = \begin{pmatrix}\frac{\partial x_1'}{\partial x_1} & \cdots & \frac{\partial x_n'}{\partial x_1}\\ \vdots & & \vdots\\ \frac{\partial x_1'}{\partial x_n} & \cdots & \frac{\partial x_n'}{\partial x_n}\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x_1'}\\ \vdots\\ \frac{\partial}{\partial x_n'}\end{pmatrix}$$

$$= T\,\nabla'$$　∴ナブラがベクトルとして変換される //

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III. III.1 $A=\begin{pmatrix}A_1\\A_2\\A_3\end{pmatrix}$ $B=\begin{pmatrix}B_1\\B_2\\B_3\end{pmatrix}$ $C=\begin{pmatrix}C_1\\C_2\\C_3\end{pmatrix}$ とする

$$A\cdot(B\times C)=A\cdot\begin{pmatrix}B_2C_3-B_3C_2\\B_3C_1-B_1C_3\\B_1C_2-B_2C_1\end{pmatrix}=\begin{matrix}A_1(B_2C_3-B_3C_2)\\+A_2(B_3C_1-B_1C_3)\\+A_3(B_1C_2-B_2C_1)\end{matrix}$$

$$B\cdot(C\times A)=B\cdot\begin{pmatrix}C_2A_3-C_3A_2\\C_3A_1-C_1A_3\\C_1A_2-C_2A_1\end{pmatrix}=\begin{matrix}B_1(C_2A_3-C_3A_2)\\+B_2(C_3A_1-C_1A_3)\\+B_3(C_1A_2-C_2A_1)\end{matrix}$$

$$C\cdot(A\times B)=C\cdot\begin{pmatrix}A_2B_3-A_3B_2\\A_3B_1-A_1B_3\\A_1B_2-A_2B_1\end{pmatrix}=\begin{matrix}C_1(A_2B_3-A_3B_2)\\+C_2(A_3B_1-A_1B_3)\\+C_3(A_1B_2-A_2B_1)\end{matrix}$$

$$\det\begin{pmatrix}A_1&B_1&C_1\\A_2&B_2&C_2\\A_3&B_3&C_3\end{pmatrix}=A_1B_2C_3+A_2B_3C_1+A_3B_1C_2-A_3B_2C_1-A_2B_1C_3-A_1B_3C_2$$

$$\therefore A\cdot(B\times C)=B\cdot(C\times A)=C\cdot(A\times B)=\det\begin{pmatrix}A_1&B_1&C_1\\A_2&B_2&C_2\\A_3&B_3&C_3\end{pmatrix}$$ //

III.2 $$(A\times B)\cdot(C\times D)=\begin{pmatrix}A_2B_3-A_3B_2\\A_3B_1-A_1B_3\\A_1B_2-A_2B_1\end{pmatrix}\cdot\begin{pmatrix}C_2D_3-C_3D_2\\C_3D_1-C_1D_3\\C_1D_2-C_2D_1\end{pmatrix}$$

$$=(A_2B_3-A_3B_2)(C_2D_3-C_3D_2)+(A_3B_1-A_1B_3)(C_3D_1-C_1D_3)+(A_1B_2-A_2B_1)(C_1D_2-C_2D_1)$$

$$(A\cdot C)(B\cdot D)-(A\cdot D)(B\cdot C)$$
$$=(A_1C_1+A_2C_2+A_3C_3)(B_1D_1+B_2D_2+B_3D_3)-(A_1D_1+A_2D_2+A_3D_3)(B_1C_1+B_2C_2+B_3C_3)$$

展開してくらべると $(A\times B)\cdot(C\times D)=(A\cdot C)(B\cdot C)-(A\cdot D)(B\cdot C)$ //

III.3　III.2より　(左辺) $= (A\times B)_i \times (C\times D)_i$

$$= \varepsilon_{iab} A_a B_b \times \varepsilon_{icd} C_c D_d$$

$$= \varepsilon_{iab}\varepsilon_{icd} A_a B_b C_c D_d$$

$$(\text{右辺}) = (\delta_{ac} A_a C_c)(\delta_{bd} B_b D_d) - (\delta_{ad} A_a D_d)(\delta_{bc} B_b C_c)$$

$$= (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) A_a B_b C_c D_d$$

$$\therefore \varepsilon_{iab}\varepsilon_{icd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$$

//

III.4　$C=A,\ D=B$ とおくと

$$|A\times B|^2 = |A|^2|B|^2 - |A\cdot B|^2$$

$$= |A|^2|B|^2 - |A|^2|B|^2\cos^2\theta$$

$$= |A|^2|B|^2 (1-\cos^2\theta$$

$$= |A|^2|B|^2 \sin^2\theta$$

$0\leq\theta\leq\pi$ より　$\sin\theta \geq 0$

$$\therefore |A\times B| = |A||B|\sin\theta$$

//

IV　IV.1　$\vec{L} = \vec{r}\times\vec{p} = m(\vec{r}\times\vec{v})$

III.4より　$|\vec{L}| = m|\vec{r}\times\vec{v}|$

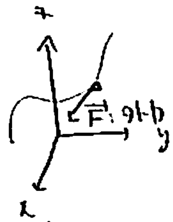
$$= m|\vec{r}||\vec{v}|\sin\frac{\pi}{2}$$

$$= m R\cdot R\omega$$

$$= mR^2\omega$$

// 外積の定義より $\vec{L}$ の向きは $\vec{r}\to\vec{p}$ へ
右ねじを回したときのねじの進む方向 //

IV.2



外力 $\vec{F} = -\frac{\vec{r}}{|\vec{r}|}|\vec{F}|$ とあらわせる

Newton eq より $m\ddot{\vec{r}} = -\frac{\vec{r}}{|\vec{r}|}|\vec{F}|$

$$\dot{\vec{p}} = -\frac{\vec{r}}{|\vec{r}|}|\vec{F}|$$

$$\vec{r}\times\dot{\vec{p}} = \vec{r}\times\left(-\frac{\vec{r}}{|\vec{r}|}|\vec{F}|\right) = -\frac{|\vec{F}|}{|\vec{r}|}\underbrace{(\vec{r}\times\vec{r})}_{=0}$$

∴ $\dot{\vec{L}} = \vec{0}$ ∴ $\vec{L} = \vec{A}$

($\vec{A}$: 任意の定数ベクトル)

∴ 題意は示された //

V. $A' = TA$, $B' = TB$ と変換されるとすると

$(A'\times B') = T(A\times B)$ を示せばよい

$$(A\times B)_i = \varepsilon_{ijk}A_jB_k$$

$$(\tilde{T}(A'\times B'))_i = \tilde{T}_{ij}(A'\times B')_j$$

$$= T_{ji}((TA)\times(TB))_j$$

$$= T_{ji}\varepsilon_{jkl}(TA)_k(TB)_l$$

$$= T_{ji}\varepsilon_{jkl}T_{km}A_mT_{ln}B_n$$

$$= \underbrace{\varepsilon_{jkl}T_{ji}T_{km}T_{ln}}_{=\ \det\tilde{T} = 1}A_mB_n$$

$$= \varepsilon_{imn}A_mB_n = \varepsilon_{ijk}A_jB_k = (A\times B)_i$$

∴ 題意は示された //