201113019
No. 張マリ
Date 2011 . 6 . 7(水)

力学A

I.1. $f(X(x,y), Y(x,y)) = \tilde{f}(x,y)$

$$\delta f = \frac{\partial f}{\partial X}\delta X + \frac{\partial f}{\partial Y}\delta Y, \quad \delta X = \frac{\partial X}{\partial x}\delta x + \frac{\partial X}{\partial y}\delta y, \quad \delta Y = \frac{\partial Y}{\partial x}\delta x + \frac{\partial Y}{\partial y}\delta y$$

$$\delta f = \frac{\partial f}{\partial X}\frac{\partial X}{\partial x}\delta x + \frac{\partial f}{\partial X}\frac{\partial X}{\partial y}\delta y + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial x}\delta x + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial y}\delta y$$

$$\frac{\delta \tilde{f}}{\delta x} = \frac{\partial \tilde{f}}{\partial X}\frac{\partial X}{\partial x} + \underbrace{\frac{\partial \tilde{f}}{\partial X}\frac{\partial X}{\partial y}\frac{\delta y}{\delta x}}_{0} + \frac{\partial \tilde{f}}{\partial Y}\frac{\partial Y}{\partial x} + \underbrace{\frac{\partial \tilde{f}}{\partial Y}\frac{\partial Y}{\partial y}\frac{\delta y}{\delta x}}_{0}$$

$\delta x \to 0 \quad \frac{\delta y}{\delta x} = 0$

$$\therefore \frac{\partial \tilde{f}}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial \tilde{f}}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial \tilde{f}}{\partial Y}$$

同様に $\therefore \frac{\partial \tilde{f}}{\partial y} = \frac{\partial X}{\partial y}\frac{\partial \tilde{f}}{\partial X} + \frac{\partial Y}{\partial y}\frac{\partial \tilde{f}}{\partial Y}$

これを $\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y}$, $\frac{\partial}{\partial y} = \frac{\partial X}{\partial y}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y}$ と書く。

I.2 $X_J\ (J=1,\cdots,m)$ の関数 $f(X_1,\cdots,X_m)$ に関して $X_J = X_J(x_1,\cdots,x_n)$ と X_J が $x_I\ (I=1,\cdots,n)$ の関数であるとき、以下を導け

$$\frac{\partial}{\partial x_I} = \frac{\partial X_J}{\partial x_I}\frac{\partial}{\partial X_J}$$

I.1より $\delta f = \sum_{J=1}^{m}\frac{\partial f}{\partial X_J}\delta X_J \qquad \delta X_J = \sum_{l=1}^{n}\frac{\partial X_J}{\partial x_l}\delta x_l$

$$\delta f = \sum_{J=1}^{m}\frac{\partial f}{\partial X_J}\sum_{l=1}^{n}\frac{\partial X_J}{\partial x_l}\delta x_l$$

$$\delta \tilde{f} = \sum_{l=1}^{n}\frac{\partial \tilde{f}}{\partial x_l}\delta x_l = \sum_{J=1}^{m}\frac{\partial f}{\partial X_J}\sum_{l=1}^{n}\frac{\partial X_J}{\partial x_l}\delta x_l$$

$$\frac{\partial f}{\partial x_l}\delta x_l = \sum_{J=1}^{m}\frac{\partial f}{\partial X_J}\frac{\partial X_J}{\partial x_l}\delta x_l$$

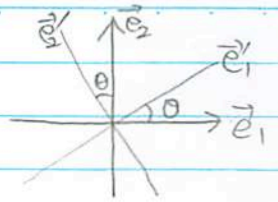
$$\therefore \frac{\partial}{\partial x_l} = \sum_{J=1}^{m}\frac{\partial}{\partial X_J}\frac{\partial X_J}{\partial x_l}$$

$$\therefore \frac{\partial}{\partial x_I} = \frac{\partial X_J}{\partial x_I}\frac{\partial}{\partial X_J}$$

No. 201113019
Date ・6・7(水)

力学A

Ⅱ.1-1



$\vec{e}_1' = \vec{e}_1\cos\theta + \vec{e}_2\sin\theta$
$\vec{e}_2' = -\vec{e}_1\sin\theta + \vec{e}_2\cos\theta$
$\vec{e}_3' = \vec{e}_3$

Ⅱ.1-2. Ⅱ1-1より，$\vec{e}_1' = \vec{e}_1\cos\theta + \vec{e}_2\sin\theta$
$\vec{e}_2' = -\vec{e}_1\sin\theta + \vec{e}_2\cos\theta$
$\vec{e}_3' = \vec{e}_3$

また 基 $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ について
$\vec{e}_i' = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\mathbb{e}_i'$ より $\mathbb{e}_1' = \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}$ $\mathbb{e}_2' = \begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}$ $\mathbb{e}_3' = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$

$\vec{v} = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\mathbb{v}$
$\vec{v}' = (\vec{e}_1', \vec{e}_2', \vec{e}_3')\mathbb{v}'$
$\vec{e}_1 = (\vec{e}_1, \vec{e}_2, \vec{e}_3)\underbrace{\mathbb{e}_1}_{\begin{pmatrix}1\\0\\0\end{pmatrix}}$ $\mathbb{e}_1 = \begin{pmatrix}e_{11}\\ e_{12}\\ e_{13}\end{pmatrix}$ とすると 同様に $\mathbb{e}_2 = \begin{pmatrix}e_{21}\\ e_{22}\\ e_{23}\end{pmatrix}$ $\mathbb{e}_3 = \begin{pmatrix}e_{31}\\ e_{32}\\ e_{33}\end{pmatrix}$

Ⅱ1-3 このとき
$T = \begin{pmatrix} e_{11}' & e_{21}' & e_{31}' \\ e_{12}' & e_{22}' & e_{32}' \\ e_{13}' & e_{23}' & e_{33}' \end{pmatrix}$ $e_i \cdot e_j = \delta_{ij} = \begin{cases} 1 & (i=j) \\ 0 & \text{それ以外} \end{cases}$

$\mathbb{e}_1' = \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}$ $\mathbb{e}_2' = \begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}$ $\mathbb{e}_3' = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$

$T = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\tilde{T} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$T\tilde{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\tilde{T} = T^{-1}$ よって Tは直交行列である。

Ⅱ.2 位置ベクトル $r = Tr'$ において
Tは直交行列であるので $\tilde{T} = T^{-1}$
$r = Tr'$ の両辺に左から T^{-1} をかけて $T^{-1}r = T^{-1}Tr'$
$T^{-1}r = r'$ $\therefore r' = \tilde{T}r$

Ⅱ.3 $r_i = x_i$
$\nabla = \begin{pmatrix}\frac{\partial}{\partial x_1}\\ \vdots \\ \frac{\partial}{\partial x_n}\end{pmatrix}$ $\nabla' = \begin{pmatrix}\frac{\partial}{\partial x_1'}\\ \vdots \\ \frac{\partial}{\partial x_n'}\end{pmatrix}$

Ⅰ.2より $\frac{\partial}{\partial x_i} = \frac{\partial x_j'}{\partial x_i}\frac{\partial}{\partial x_j'} = \frac{\partial x_1'}{\partial x_i}\frac{\partial}{\partial x_1'} + \frac{\partial x_2'}{\partial x_i}\frac{\partial}{\partial x_2'} + \cdots + \frac{\partial x_n'}{\partial x_i}\frac{\partial}{\partial x_n'}$

$= \begin{pmatrix}\frac{\partial x_1'}{\partial x_i} & \frac{\partial x_2'}{\partial x_i} & \cdots & \frac{\partial x_n'}{\partial x_i}\end{pmatrix} \begin{pmatrix}\frac{\partial}{\partial x_1'}\\ \frac{\partial}{\partial x_2'}\\ \vdots \\ \frac{\partial}{\partial x_n'}\end{pmatrix}$

No. CHANG MARI

Date . 6 . 7

力学A

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1'}{\partial x_1} & \cdots & \frac{\partial x_n'}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_1'}{\partial x_n} & \cdots & \frac{\partial x_n'}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1'} \\ \vdots \\ \frac{\partial}{\partial x_n'} \end{pmatrix}$$

$$= T\nabla'$$

$\therefore \nabla = T\nabla'$となり、ナブラがベクトルとして変換される。

III.1 $A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B) = \det\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$

$$B\times C \Rightarrow \begin{vmatrix} B_1 & B_2 & B_3 & B_1 \\ C_1 & C_2 & C_3 & C_1 \end{vmatrix} = \begin{pmatrix} B_2C_3 - B_3C_2 \\ B_3C_1 - B_1C_3 \\ B_1C_2 - B_2C_1 \end{pmatrix}$$

$$A\cdot(B\times C) = A_1B_2C_3 - A_1B_3C_2 + A_2B_3C_1 - A_2B_1C_3 + A_3B_1C_2 - A_3B_2C_1$$

$$B\cdot(C\times A) = B_1C_2A_3 - B_1C_3A_2 + B_2C_3A_1 - B_2C_1A_3 + B_3C_1A_2 - B_3C_2A_1$$

$$C\cdot(A\times B) = C_1A_2B_3 - C_1A_3B_2 + C_2A_3B_1 - C_2A_1B_3 + C_3A_1B_2 - C_3A_2B_1$$

$$\det\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix} = A_1B_2C_3 + A_2B_3C_1 + B_1C_2A_3 - C_1B_2A_3 - B_1A_2C_3 - A_1C_2B_3$$

$$\therefore A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B) = \det\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$$

III.2 $(A\times B)\cdot(C\times D) = (A\cdot C)(B\cdot D) - (A\cdot D)(B\cdot C)$

$$A\times B = \begin{pmatrix} A_2B_3 - A_3B_2 \\ A_3B_1 - A_1B_3 \\ A_1B_2 - A_2B_1 \end{pmatrix} \quad C\times D = \begin{pmatrix} C_2D_3 - C_3D_2 \\ C_3D_1 - C_1D_3 \\ C_1D_2 - C_2D_1 \end{pmatrix}$$

$$(A\times B)\cdot(C\times D) = (A_2B_3 - A_3B_2)(C_2D_3 - C_3D_2) + (A_3B_1 - A_1B_3)(C_3D_1 - C_1D_3) + (A_1B_2 - A_2B_1)(C_1D_2 - C_2D_1)$$

$$(A\cdot C)(B\cdot D) - (A\cdot D)(B\cdot C) = (A_1C_1 + A_2C_2 + A_3C_3)(B_1D_1 + B_2D_2 + B_3D_3) - (A_1D_1 + A_2D_2 + A_3D_3)(B_1C_1 + B_2C_2 + B_2C_3)$$

まじめに計算すると

$\therefore (A\times B)\cdot(C\times D) = (A\cdot C)(B\cdot D) - (A\cdot D)(B\cdot C)$となる。

III.3 $(A\times B)_i = \varepsilon_{iab}A_aB_b$

III.2より $\varepsilon_{iab}\varepsilon_{icd}A_aB_bC_cD_d = A_aC_aB_bD_b - A_aD_aB_bC_b$ となる。

$A_aC_aB_bD_b = \delta_{ac}\delta_{bd}A_aB_bC_cD_d$, $A_aD_aB_bC_b = \delta_{ad}\delta_{bc}A_aB_bC_cD_d$

$$\varepsilon_{iab}\varepsilon_{icd}A_aB_bC_cD_d = \delta_{ac}\delta_{bd}A_aB_bC_cD_d - \delta_{ad}\delta_{bc}A_aB_bC_cD_d$$

$\therefore \varepsilon_{iab}\varepsilon_{icd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$ となる。

No. 201113019

Date ・6・7

力学A

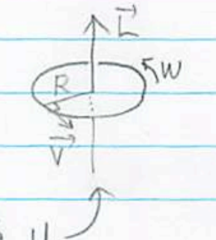
Ⅲ4. $\mathbb{A}=\mathbb{C}$, $\mathbb{B}=\mathbb{D}$ とする [Ⅲ.2]から

$$(\mathbb{A}\times\mathbb{B})\cdot(\mathbb{A}\times\mathbb{B}) = (\mathbb{A}\cdot\mathbb{A})(\mathbb{B}\cdot\mathbb{B}) - (\mathbb{A}\cdot\mathbb{B})(\mathbb{A}\cdot\mathbb{B})$$

$$|\mathbb{A}\times\mathbb{B}|^2 = |\mathbb{A}|^2|\mathbb{B}|^2 - |\mathbb{A}\cdot\mathbb{B}|^2$$
$$= |\mathbb{A}|^2|\mathbb{B}|^2 - |\mathbb{A}|^2|\mathbb{B}|^2\cos^2\theta$$
$$= |\mathbb{A}|^2|\mathbb{B}|^2(1-\cos^2\theta)$$
$$= |\mathbb{A}|^2|\mathbb{B}|^2\sin^2\theta$$

$\therefore\ |\mathbb{A}\times\mathbb{B}| = |\mathbb{A}||\mathbb{B}|\sin\theta$

Ⅳ1. 角運動量 $\vec{L}$は $\vec{L}=\vec{r}\times\vec{p}$



大きさ$|\vec{L}|$は $|\vec{L}|=|\vec{r}|\times|\vec{p}|$

ここで $|\vec{r}|=R$

$|\vec{p}|=mv=m\omega R$ より

$|\vec{L}|=m\omega R^2$ 向きは図の通り

Ⅳ2. $\vec{L}=\vec{r}\times m\frac{d}{dt}\vec{r}$

両辺をtで微分すると,

$$\frac{d}{dt}\vec{L}=\underbrace{\left(\frac{d}{dt}\vec{r}\times m\frac{d}{dt}\vec{r}\right)}_{}+\left(\vec{r}\times m\frac{d^2}{dt^2}\vec{r}\right)$$

→平行なベクトルの外積で0となる

また、$m\frac{d^2}{dt^2}\vec{r}$ は質点mにかかる力であり

これは原点を向いており, $F \parallel m\frac{d^2}{dt^2}\vec{r}$ となるから

結局第2項も0となり

$\frac{d}{dt}\vec{L}=0$となる よって角運動量は保存している。

Ⅴ. $A'=TA$ $B'=TB$

$$\tilde{T}(A'\times B')_I=\sum_{\substack{ijk\\JK}}\tilde{T}_{Ii}(\varepsilon_{ijk}T_{jJ}T_{kK}A_JB_K)$$

$$=\sum_{\substack{ijk\\JK}}\varepsilon_{ijk}T_{iI}T_{jJ}T_{kK}A_JB_K$$

$$\det T=\sum_{ijk}\varepsilon_{ijk}T_{1i}T_{2j}T_{3k}$$

IJKの並びによってijkの並びも変化するから

$$\tilde{T}(A'\times B')_I=\sum_{JK}\varepsilon_{IJK}\underbrace{\det\tilde{T}}_{=1}A_JB_K$$

$$=\sum_{JK}\varepsilon_{IJK}A_JB_K=(A\times B)_I$$

$\therefore\ A'\times B'=T(A\times B)$