数学２C (水曜２限、10:15　11:45）第六回レポート解答例

1.　$f(z) = \dfrac{z}{(2z+1)(z-2)}$ として次の Taylor 展開、または Laurent 展開を求めよ.

まず、

$$f(z) = \frac{z}{(2z+1)(z-2)} \quad = \quad \frac{1}{5}\left(\frac{1}{1+2z} - \frac{1}{1-\frac{z}{2}}\right) \tag{1}$$

と部分分数分解できる。

また、$\dfrac{1}{1-x}$, $\dfrac{1}{1+x}$ の $x=0$ 周りでの Taylor 展開は以下のようになる（$x \leq 1$)。

$$\frac{1}{1-x} \quad = \quad 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \tag{2}$$

$$\frac{1}{1+x} \quad = \quad 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \tag{3}$$

i).　$z = 0$ まわりで $|z| \leq \dfrac{1}{2}$, $\dfrac{1}{2} \leq |z| \leq 2$, $2 \leq |z|$ の各場合.

・$|z| \leq \dfrac{1}{2}$ のとき

$|2z| \leq 1$, $|\dfrac{z}{2}| \leq 1$ より、

$$\frac{1}{1+2z} \quad = \quad 1 + (-2)z + (-2)^2 z^2 + \cdots + (-2)^n z^n + \cdots = \sum_{k=0}^{\infty} (-2)^k z^k \tag{4}$$

$$\frac{1}{1-\frac{z}{2}} \quad = \quad 1 + \left(\frac{1}{2}\right) z + \left(\frac{1}{2}\right)^2 z^2 + \cdots + \left(\frac{1}{2}\right)^n z^n + \cdots = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^k \tag{5}$$

よって、

$$f(z) \quad = \quad \frac{1}{5}\left(\sum_{k=0}^{\infty} (-2)^k z^k - \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^k\right) \tag{6}$$

$$\quad = \quad \frac{1}{5} \sum_{k=0}^{\infty} \left((-2)^k - \left(\frac{1}{2}\right)^k\right) z^k \tag{7}$$

・$\dfrac{1}{2} \leq |z| \leq 2$ のとき

$|\dfrac{1}{2z}| \leq 1,\ |\dfrac{z}{2}| \leq 1$ より、

$$\frac{1}{1+2z} = \frac{1}{2z}\frac{1}{1+\frac{1}{2z}} \tag{8}$$

$$= \frac{1}{2z}\left(1+\left(-\frac{1}{2}\right)\frac{1}{z}+\left(-\frac{1}{2}\right)^2\frac{1}{z^2}+\cdots+\left(-\frac{1}{2}\right)^n\frac{1}{z^n}+\cdots\right) \tag{9}$$

$$= \frac{1}{2z}\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k\frac{1}{z^k} = (-1)\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^{k+1}\frac{1}{z^{k+1}} \tag{10}$$

$$\frac{1}{1-\frac{z}{2}} = \sum_{k=0}^{\infty}(\frac{1}{2})^k z^k \tag{11}$$

よって、

$$f(z) = \frac{1}{5}\left(-\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^{k+1}\frac{1}{z^{k+1}}-\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k z^k\right) \tag{12}$$

$$= -\frac{1}{5}\sum_{k=0}^{\infty}\left(\left(-\frac{1}{2}\right)^{k+1}\frac{1}{z^{k+1}}+\left(\frac{1}{2}\right)^k z^k\right) \tag{13}$$

・$2 \leq |z|$ のとき

$|\dfrac{1}{2z}| \leq 1,\ |\dfrac{2}{z}| \leq 1$ より、

$$\frac{1}{1+2z} = \frac{1}{2z}\frac{1}{1+\frac{1}{2z}} = (-1)\sum_{k=0}^{\infty}(-\frac{1}{2})^{k+1}\frac{1}{z^{k+1}} \tag{14}$$

$$\frac{1}{1-\frac{z}{2}} = -\frac{z}{2}\ \frac{1}{1-\frac{2}{z}} \tag{15}$$

$$= -\frac{2}{z}\left(1+2\frac{1}{z}+2^2\frac{1}{z^2}+\cdots+2^n\frac{1}{z^n}+\cdots\right) \tag{16}$$

$$= -\frac{2}{z}\sum_{k=0}^{\infty}2^k\frac{1}{z^k} = (-1)\sum_{k=0}^{\infty}2^{k+1}\frac{1}{z^{k+1}} \tag{17}$$

よって、

$$f(z) = \frac{1}{5}\left(-\sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^{k+1}\frac{1}{z^{k+1}}+\sum_{k=0}^{\infty}2^{k+1}\frac{1}{z^{k+1}}\right) \tag{18}$$

$$= -\frac{1}{5}\sum_{k=0}^{\infty}\left(\left(-\frac{1}{2}\right)^{k+1}-2^{k+1}\right)\frac{1}{z^{k+1}} \tag{19}$$

ii). $z=-\frac{1}{2}$ まわりで $|z+\frac{1}{2}|\le\frac{5}{2}$, $|z+\frac{1}{2}|\ge\frac{5}{2}$ の各場合.

$$\begin{aligned} f(z) &= \frac{z}{(2z+1)(z-2)} = \frac{1}{5}\,\frac{1}{1+2z} - \frac{1}{5}\frac{1}{1-\frac{z}{2}} && (20)\\ &= \frac{1}{10}\,\frac{1}{z+\frac{1}{2}} - \frac{4}{25}\,\frac{1}{1-\frac{2}{5}\left(z+\frac{1}{2}\right)} && (21)\end{aligned}$$

・$|z+\frac{1}{2}|\le\frac{5}{2}$ のとき

$\left|\frac{2}{5}\left(z+\frac{1}{2}\right)\right|\le 1$ より

$$\begin{aligned} \frac{1}{1-\frac{2}{5}\left(z+\frac{1}{2}\right)} &= 1+\frac{2}{5}\left(z+\frac{1}{2}\right)+\left(\frac{2}{5}\right)^2\left(z+\frac{1}{2}\right)^2+\cdots && (22)\\ &= \sum_{k=0}^{\infty}\left(\frac{2}{5}\right)^k\left(z+\frac{1}{2}\right)^k && (23)\end{aligned}$$

よって、

$$\begin{aligned} f(z) &= \frac{1}{10}\,\frac{1}{z+\frac{1}{2}} - \frac{4}{25}\sum_{k=0}^{\infty}\left(\frac{2}{5}\right)^k\left(z+\frac{1}{2}\right)^k && (24)\\ &= \frac{1}{10}\left(z+\frac{1}{2}\right)^{-1} - \sum_{k=0}^{\infty}\left(\frac{2}{5}\right)^{k+2}\left(z+\frac{1}{2}\right)^k && (25)\end{aligned}$$

・$|z+\frac{1}{2}|\ge\frac{5}{2}$ のとき

$\left|\frac{5}{2}\,\frac{1}{z+\frac{1}{2}}\right|\le 1$ より

$$\begin{aligned} \frac{1}{1-\frac{2}{5}\left(z+\frac{1}{2}\right)} &= -\frac{5}{2}\frac{1}{z+\frac{1}{2}}\frac{1}{1-\frac{5}{2}\frac{1}{z+\frac{1}{2}}} && (26)\\ &= -\frac{5}{2}\frac{1}{z+\frac{1}{2}}\left(1+\frac{5}{2}\frac{1}{z+\frac{1}{2}}+\left(\frac{5}{2}\right)^2\frac{1}{(z+\frac{1}{2})^2}+\cdots\right) && (27)\\ &= -\frac{5}{2}\frac{1}{z+\frac{1}{2}}\sum_{k=0}^{\infty}\left(\frac{5}{2}\right)^k\frac{1}{(z+\frac{1}{2})^k} && (28)\\ &= (-1)\sum_{k=0}^{\infty}\left(\frac{5}{2}\right)^{k+1}\frac{1}{(z+\frac{1}{2})^{k+1}} && (29)\end{aligned}$$

よって、

$$\begin{aligned} f(z) &= \frac{1}{10}\,\frac{1}{z+\frac{1}{2}}+\frac{4}{25}\sum_{k=0}^{\infty}\left(\frac{5}{2}\right)^{k+1}\frac{1}{(z+\frac{1}{2})^{k+1}} && (30)\\ &= \frac{1}{10}\left(z+\frac{1}{2}\right)^{-1}+\sum_{k=0}^{\infty}\left(\frac{5}{2}\right)^{k-1}\left(z+\frac{1}{2}\right)^{-(k+1)} && (31)\\ &= \frac{1}{10}\left(z+\frac{1}{2}\right)^{-1}+\frac{2}{5}\left(z+\frac{1}{2}\right)^{-1}+\sum_{k=1}^{\infty}\left(\frac{5}{2}\right)^{k-1}\left(z+\frac{1}{2}\right)^{-(k+1)} && (32)\\ &= \frac{1}{2}\left(z+\frac{1}{2}\right)^{-1}+\sum_{k=0}^{\infty}\left(\frac{5}{2}\right)^{k}\left(z+\frac{1}{2}\right)^{-(k+2)} && (33)\end{aligned}$$

2. $f(z) = \dfrac{\sin \pi z}{z-1}$ を $z=1$ まわりで Laurent 展開せよ.

$\sin \pi z$ の $z=1$ まわりの Taylar 展開は次のようになる。

$$\sin \pi z = -\pi(z-1) + \frac{\pi^3}{3!}(z-1)^3 - \frac{\pi^5}{5!}(z-1)^5 + \cdots \tag{34}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{(k+1)}\pi^{2k+1}}{(2k+1)!}(z-1)^{2k+1} \tag{35}$$

よって、

$$f(z) = \frac{1}{z-1}\sin \pi z \tag{36}$$

$$= \frac{1}{z-1}\sum_{k=0}^{\infty} \frac{(-1)^{(k+1)}\pi^{2k+1}}{(2k+1)!}(z-1)^{2k+1} \tag{37}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{(k+1)}\pi^{2k+1}}{(2k+1)!}(z-1)^{2k} \tag{38}$$