Time-Reversal Invariance & Edge States of Topological Insulators

Physics of the Bulk-Edge correspondence

時間反転対称性とトポロジカル絶縁体のエッジ状態

バルクエッジ対応の物理

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筑波大学大学院数理物質科学研究科物理学専攻

初貝安弘

- Classical to Quantum
  - More than Moore to the real breakthrough
  - Use of quantum coherence
  - Topological insulator (Quantum Spin Hall effect): From Spin to Spinor
  - Time-reversal
  - Kramers degeneracy
  - Rotation: Spin & spinor

Classical & Quantum observables

- "Classical" Observables

Unitary invariant

  - Charge density, Spin density,...

$$\mathcal{O} = n_\uparrow \pm n_\downarrow, \cdots$$

$$\langle \mathcal{O} \rangle_G = \langle G|\mathcal{O}|G\rangle = \langle G'|\mathcal{O}|G'\rangle = \langle \mathcal{O} \rangle_{G'}$$

charge, magnetization, ...

$$|G'\rangle = |G\rangle e^{i\phi}$$

- "Quantum" Observables !

depend on the phase

  - Quantum Interferenc

polarization

$$G'_1|G'_2\rangle e^{i(\phi_1-\phi_2)}$$

$$|G_i\rangle = |G'_i\rangle e^{i\phi_i}$$

    - Diffraction
    - Aharonov-Bohm Effect
    - Berry phases

$$\langle G|G +$$

Hall conductance

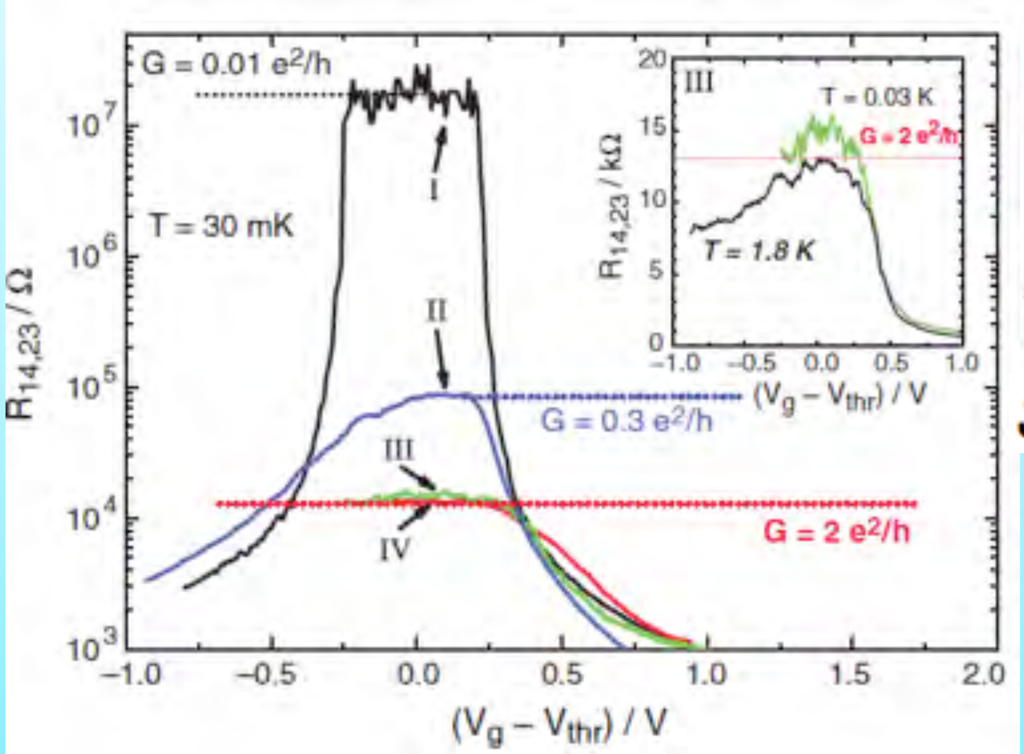
$$A = \langle G|dG\rangle$$:Berry Connection

$$i\gamma = \int A$$:Berry Phase

$$C = \frac{1}{2\pi i}\int dA$$:Chern number

Topological Insulator

Topological insulator : Quantum Spin Hall state



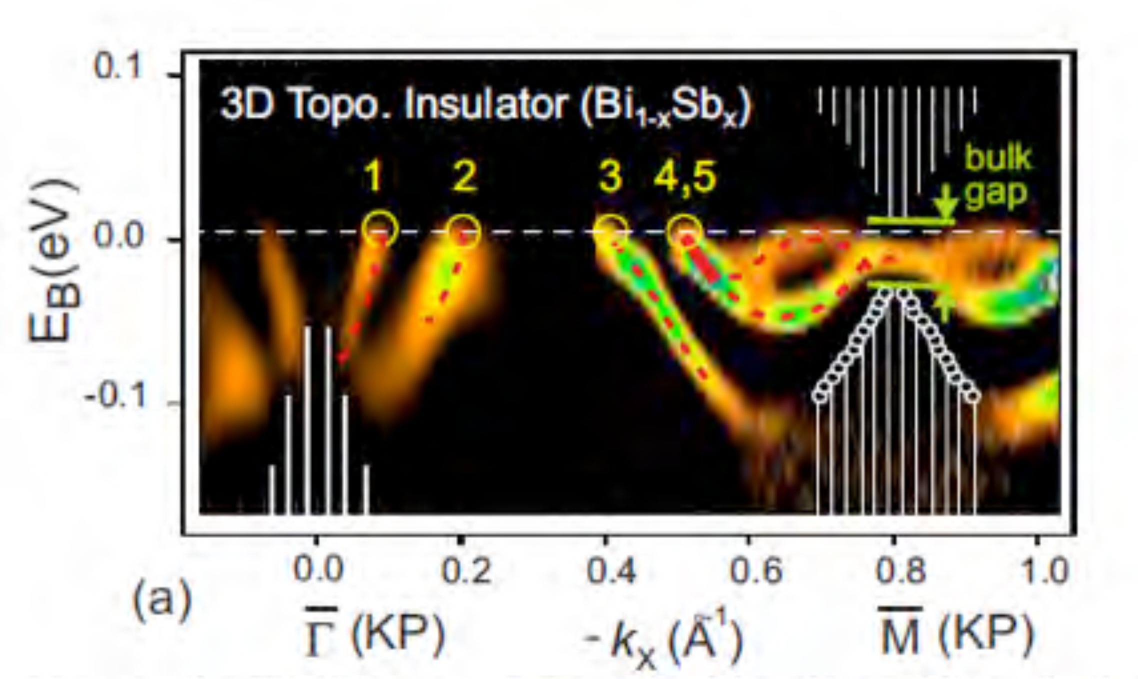
Quantum Spin Hall Insulator State in HgTe Quantum Wells

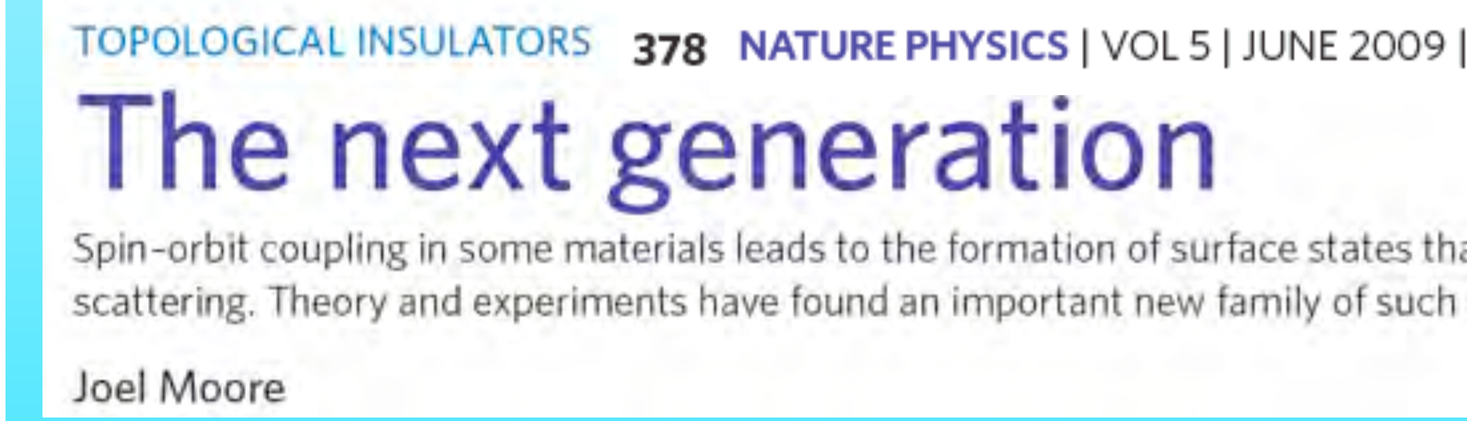
Markus König,[1] Steffen Wiedmann,[1] Christoph Brüne,[1] Andreas Roth,[1] Hartmut Buhmann,[1] Laurens W. Molenkamp,[1*] Xiao-Liang Qi,[2] Shou-Cheng Zhang[2]

Science **318**, 766 (2007)

2D

TNG





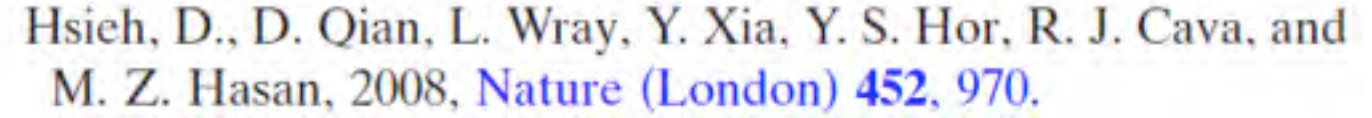
TOPOLOGICAL INSULATORS **378 NATURE PHYSICS** | VOL 5 | JUNE 2009 |

The next generation

Spin-orbit coupling in some materials leads to the formation of surface states tha
scattering. Theory and experiments have found an important new family of such

Joel Moore

3D



Hsieh, D., D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M. Z. Hasan, 2008, Nature (London) **452**, 970.

Classical to Quantum

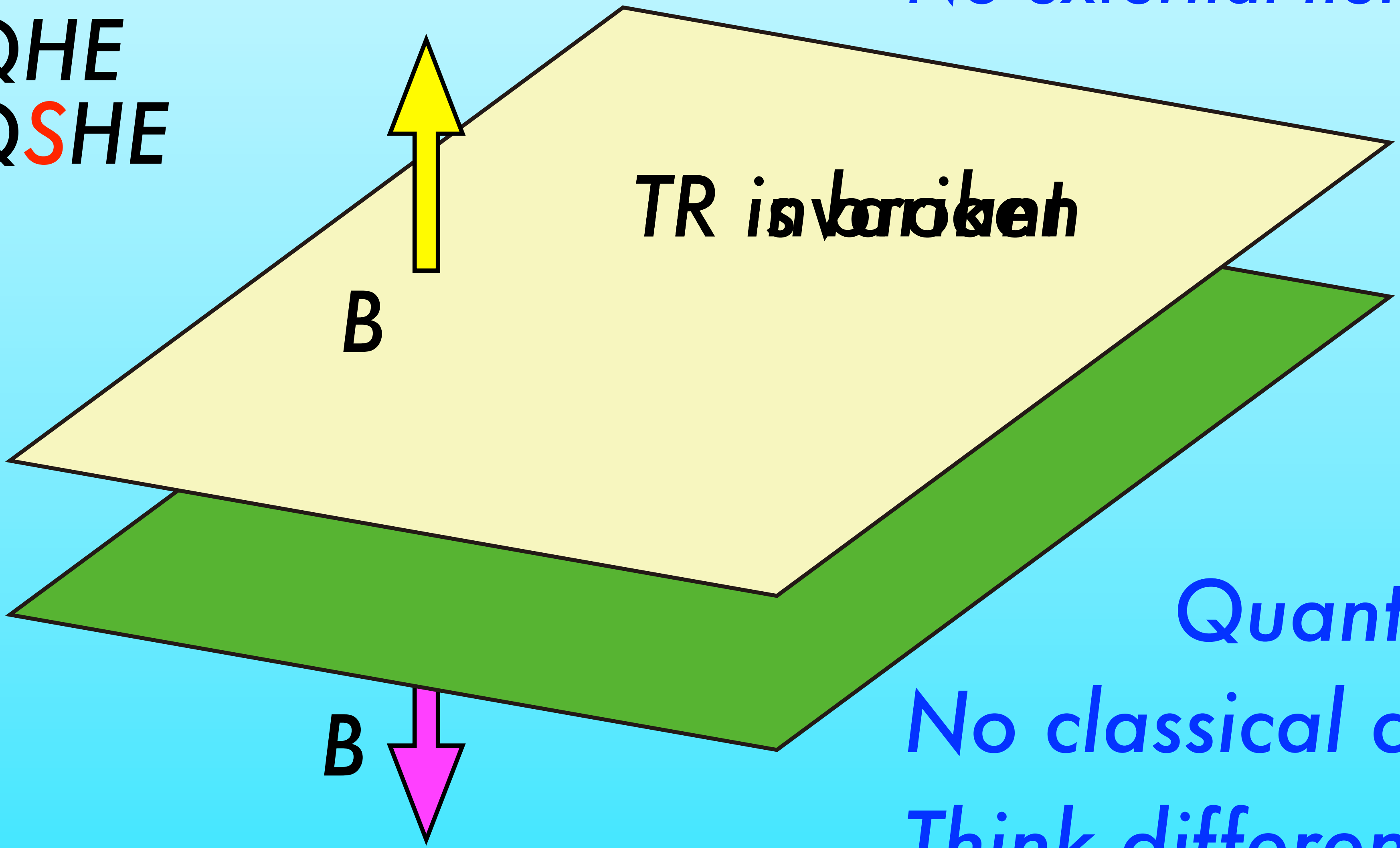
"Quantum" Spin Hall Effect

= Quantum Hall Effect without magnetic field

= Quantum Hall Effect with time-reversal invariance

No external net magnetic field

QHE
QSHE



spin up

spin down

Quantum effects !

No classical correspondent !

Think different

Following the QHE

so-called Topological Insulator

Topological insulator : Quantum Spin Hall state

Need to undestand !

Time Reversal

Kramers degeneracy

Let me explain !

Spin Hall conductance is not quantized

Spin is not conserved (spin-orbit)

Time-Reversal (TR) symmetry & Kramers degeneracy

TR: Anti-Unitary Θ : $c_i = \begin{bmatrix} c_{i\uparrow} \\ c_{i\downarrow} \end{bmatrix} \to \begin{bmatrix} c_{i\downarrow} \\ -c_{i\uparrow} \end{bmatrix} = Jc_i$

$$\mathcal{H} = c_i^\dagger H_{ij} c_j$$

& complex conjugate

$$J = i\sigma_y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

TR invariance

$$\boxed{[\Theta, \mathcal{H}] = 0} \longrightarrow JH^*J^{-1} = H \qquad \{H\}_{ij} = H_{ij}$$

$$\overset{J}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \overset{H\,*}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^*} \overset{J^{-1}}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} c^* & d^* \\ -a^* & -b^* \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix} = \underset{H}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

$$a = d^*, \; b = -c^*$$

$$H = \begin{bmatrix} t & \Delta \\ -\Delta^* & t^* \end{bmatrix}$$

$t^\dagger = t$ hermite

t : Spin independent hopping

& hermiticity $\Delta^\dagger = -\Delta^* \to \tilde{\Delta} = -\Delta$

Δ : Spin-orbit, Rashba term, etc

anti-symmetric

Time-Reversal (TR) symmetry & Kramers degeneracy

Schrödinger Equation

$$\begin{bmatrix} t & \Delta \\ -\Delta^* & t^* \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = E\begin{bmatrix} u \\ v \end{bmatrix}$$

Kramers degeneracy

Any one particle state is doubly degenerate

↓

$$tu + \Delta v = Eu$$
$$-\Delta^* u + t^* v = Ev$$

$$tv^* + \Delta(-u^*) = Ev^*$$
$$-\Delta^* v^* + t^*(-u^*) = E(-u^*)$$

Ah !

$$H\begin{bmatrix} v^* \\ -u^* \end{bmatrix} = E\begin{bmatrix} v^* \\ -u^* \end{bmatrix}, \quad \begin{bmatrix} v^* \\ -u^* \end{bmatrix}$$ is also an eigen state with the same energy

→ $\begin{bmatrix} u_\Theta \\ v_\Theta \end{bmatrix} = \begin{bmatrix} v^* \\ -u^* \end{bmatrix}$ & $\begin{bmatrix} u \\ v \end{bmatrix}$: the same energy, degenerate ?

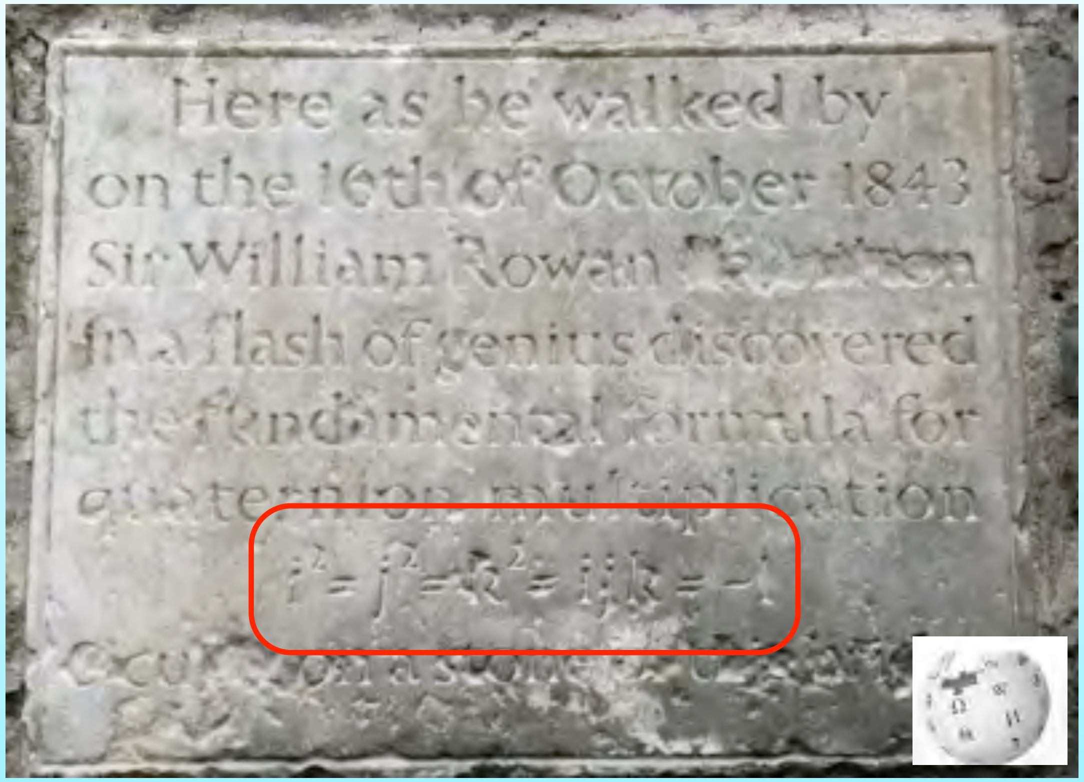
Not yet !

the same state ??

OK ! surely different !

Orthogonal ! $\begin{bmatrix} u \\ v \end{bmatrix}^\dagger \begin{bmatrix} u_\Theta \\ v_\Theta \end{bmatrix} = u^* v^* + v^*(-u^*) = 0$

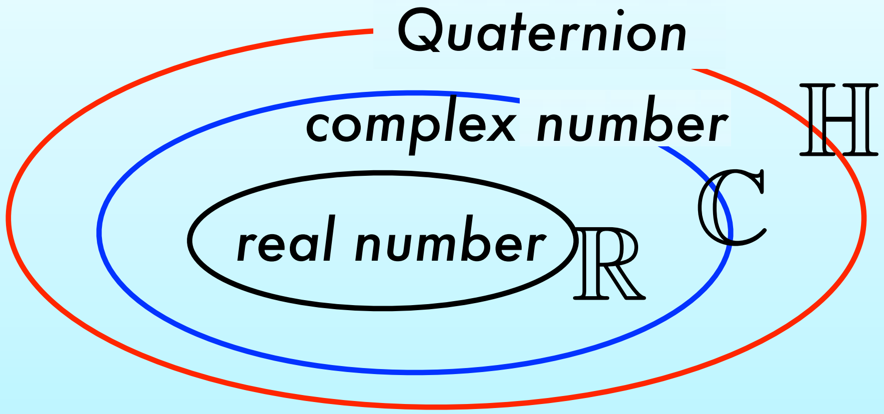
Time Reversal & Quaternions



Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

Hamilton 四元数発見の碑

F.J.Dyson '61–



No magic, neither crazy

just Pauli matrices

$$H = \begin{bmatrix} t & \Delta \\ -\Delta^* & t^* \end{bmatrix} = (\mathrm{Re}t) I_2 + (\mathrm{Im}t)\, i\sigma_z + (\mathrm{Re}\Delta)\, i\sigma_y + (\mathrm{Im}\Delta)\, i\sigma_x$$

$$\cong (\mathrm{Re}t)\, 1 + (\mathrm{Im}t) i_{\mathbb{H}} + (\mathrm{Re}\Delta)\, j_{\mathbb{H}} + (\mathrm{Im}\Delta)\, k_{\mathbb{H}}$$

: **Quaternion (四元数)**

$$i_{\mathbb{H}} \cong i\sigma_z, \;\; j_{\mathbb{H}} \cong i\sigma_y, \;\; k_{\mathbb{H}} \cong i\sigma_z$$
$$i_{\mathbb{H}}^2 = k_{\mathbb{H}}^2 = k_{\mathbb{H}}^2 = i_{\mathbb{H}} j_{\mathbb{H}} k_{\mathbb{H}} = -1$$

Quaternion 2×2 Matrix, Yang Monopole & quantization: YH, NJP12, 065004 (2010)

Time-Reversal, Spins & Spinors

spin

spinor

$$\hat{\boldsymbol{S}} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \boldsymbol{c}^\dagger \boldsymbol{S} \boldsymbol{c} \,, \quad \boldsymbol{S} = \frac{\boldsymbol{\sigma}}{2} \,, \quad \boldsymbol{c} = \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}$$

$$\Theta \hat{\boldsymbol{S}} \Theta^{-1} = \boldsymbol{c}^\dagger \boldsymbol{S}^\Theta \boldsymbol{c}$$

$$\boldsymbol{S}^\Theta = \boldsymbol{J} \boldsymbol{S}^* \boldsymbol{J}^{-1}$$

$$\Theta\, \boldsymbol{c} \Theta^{-1} = \boldsymbol{J} \boldsymbol{c}$$

$$\sigma_x^\Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\sigma_x$$

$$\sigma_y^\Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\sigma_y$$

$$\sigma_z^\Theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\sigma_z$$

$$\boldsymbol{S}^\Theta = -\boldsymbol{S}$$

$$\boxed{\boldsymbol{B} \cdot \boldsymbol{S} \rightarrow -\boldsymbol{B} \cdot \boldsymbol{S}}$$

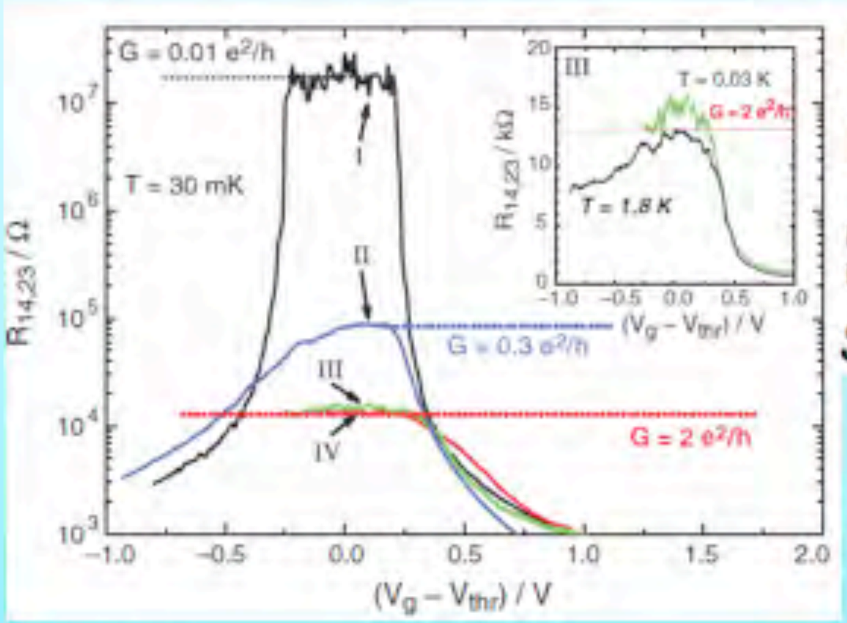
Magnetic field

Zeeman term breaks TR

Classical to Quantum

Quantum Spin Hall effect ??

Topological insulator : Quantum "Spinor" Hall state



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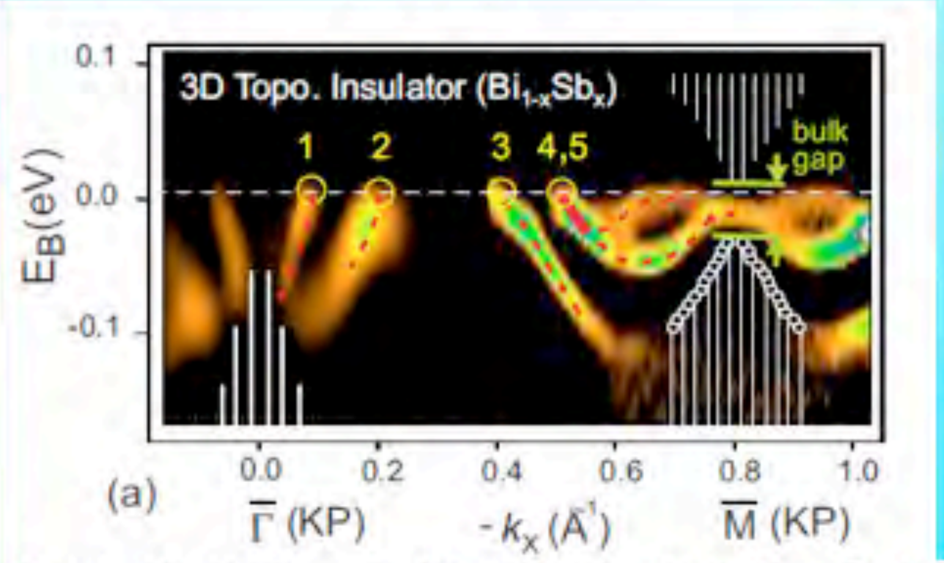

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The next generation

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Purely quantum mechanical !

- Edge is topological
  - Right / left to the symmetry : Topological phases
  - Topological protection
  - Zoo of boundary states
  - Bulk-Edge correspondence
  - Edge states of topological insulators

What's topological ?

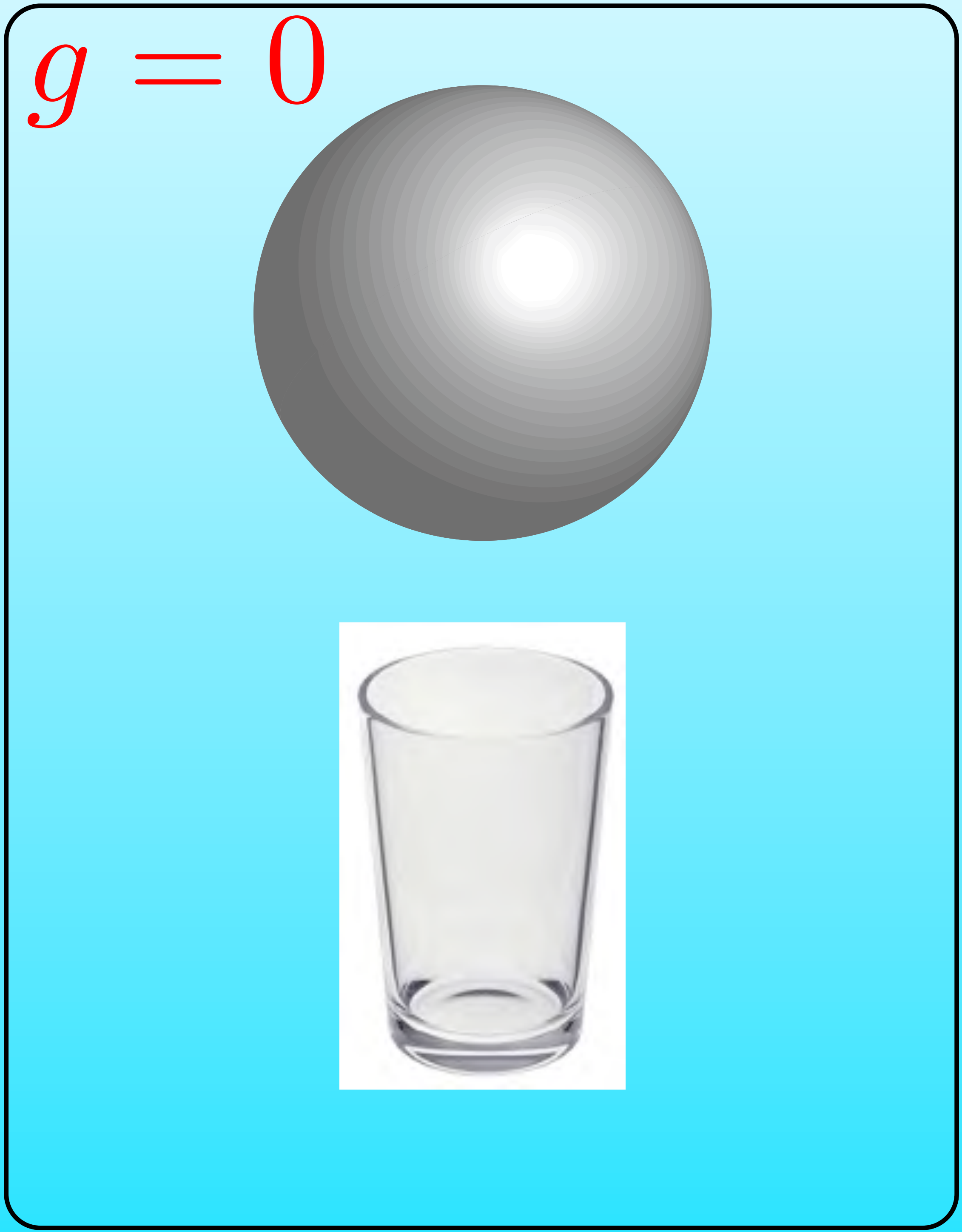
Edge is topological

Topological ?

g: # of holes

Integer!

Topological numbers

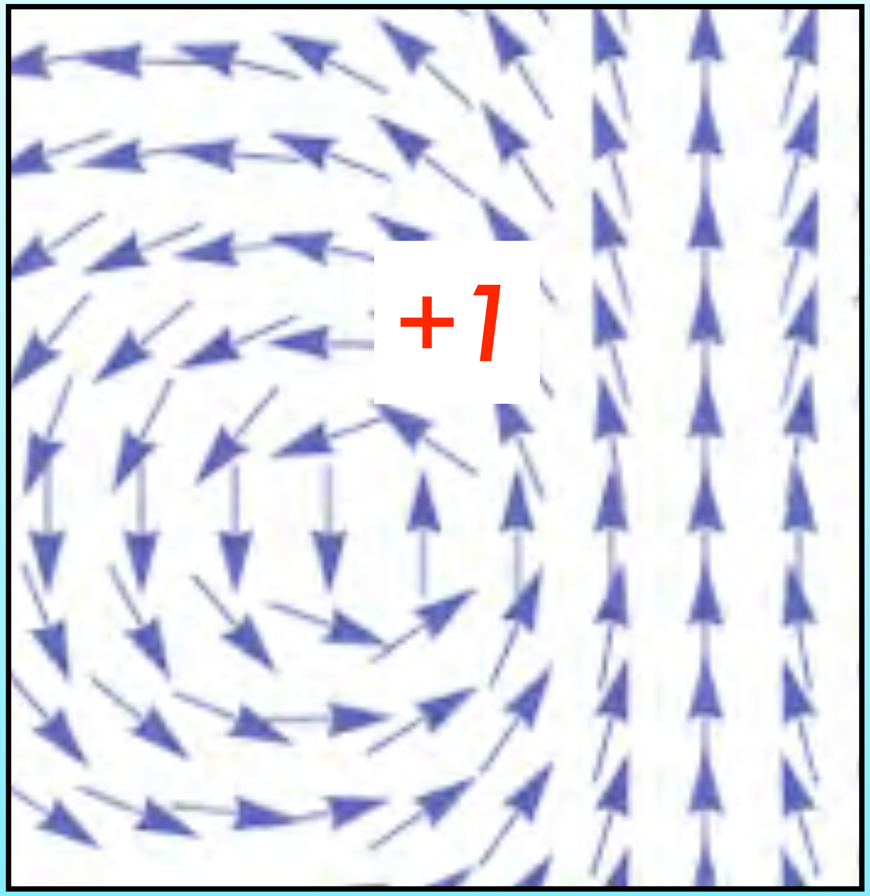


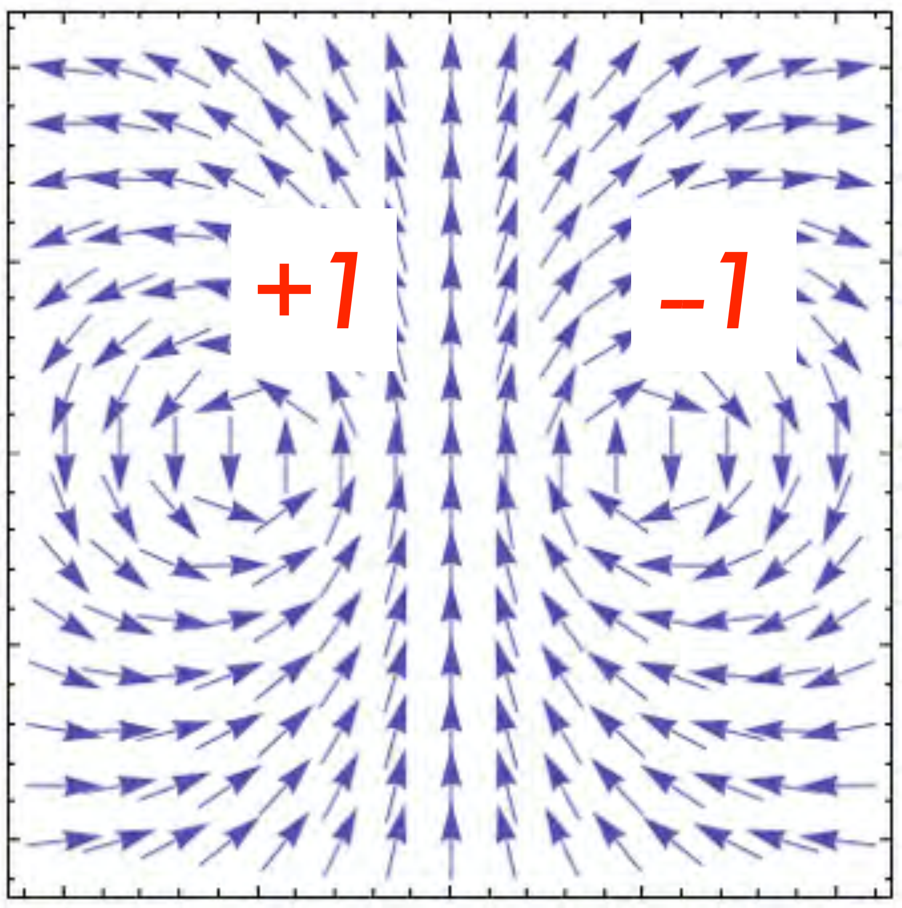
Edge is topological

Topological numbers in physics

Historical example Vortexes

stable





+1-1=0

unstable

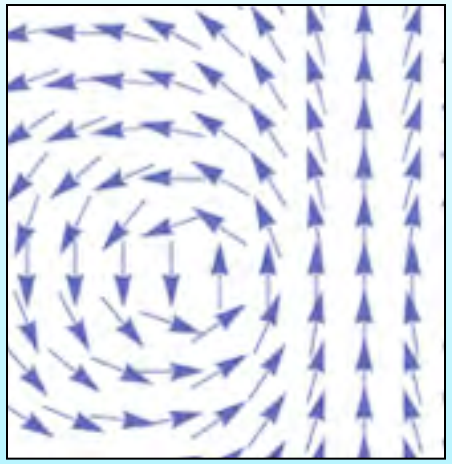
If # of the net Vortexes is finite, it's hard to disappear

Need finite energy to collapse, otherwise stays FOREVER !

Edge is topological

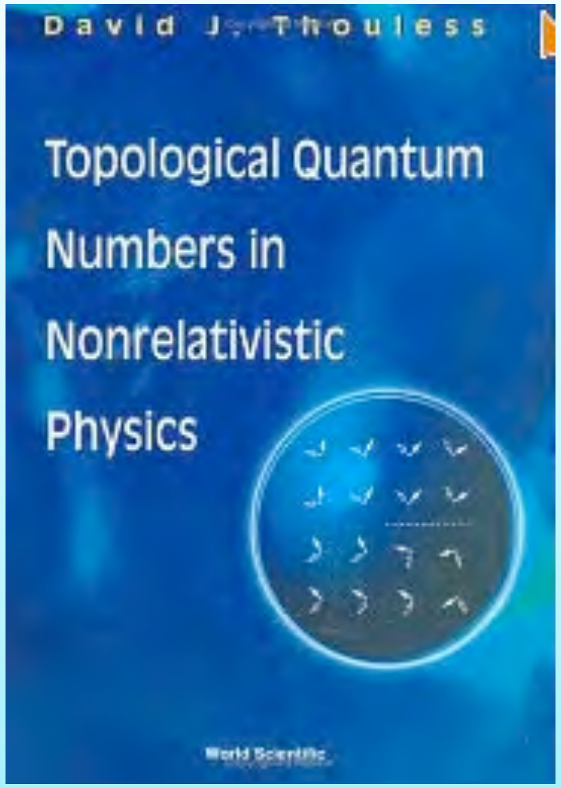
Topological stability

stable +1



Need finite energy to collapse,
otherwise stay FOREVER !

Topological stability



David J. Thouless

Topological Quantum Numbers in Nonrelativistic Physics

World Scientific

Topological number is discrete in many cases.

Quantization

superconductivity Φ_0
flux quantum

quantum Hall effect σ_{xy}

It implies possible stable devices
with low error rate !!

Topological quantum computer !!??

Not by 5/2 FQH state, now by topological insulators (?)

Edge is topological

Right / left to the symmetry

With translation invariance

$$[H,T]=0$$

Bloch theorem $T\psi(\boldsymbol{r})\equiv\psi(\boldsymbol{r}+\boldsymbol{t})=e^{ik}\psi(\boldsymbol{r})$

$$|\psi(\boldsymbol{r})|=|\psi(\boldsymbol{r}+\boldsymbol{t})|=|\psi(\boldsymbol{r}+2\boldsymbol{t})|=\cdots=|\psi(\boldsymbol{r}+10^{10}\boldsymbol{t})|=\cdots$$

Extended over the whole space

Energy bands : energy of the extended states

With boundaries/ impurities

As for extended states, effects of edges can be negligible !

dimension is less ! 0D impurities/1D boundaries in 2D

→ States in the energy gap are localized !

since they can not be extended

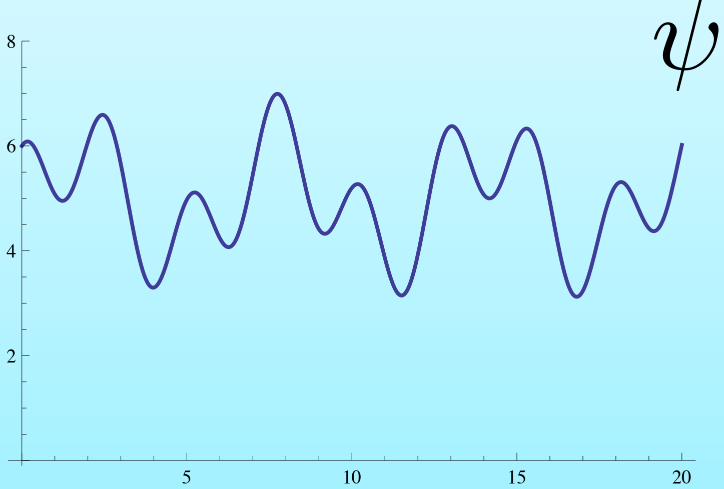
Bound states / Edge states

Edge is topological

Right / left to the symmetry

Extended states

unnormalizable

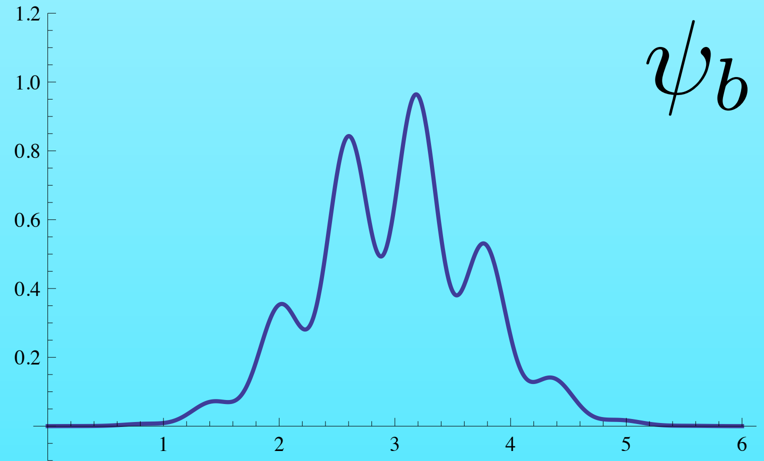


$$\psi_e(r) \sim \frac{1}{\sqrt{V}} e^{ikr} \longrightarrow 0 \quad (V \to \infty)$$

V: Volume

Bound states / Edge states

normalizable



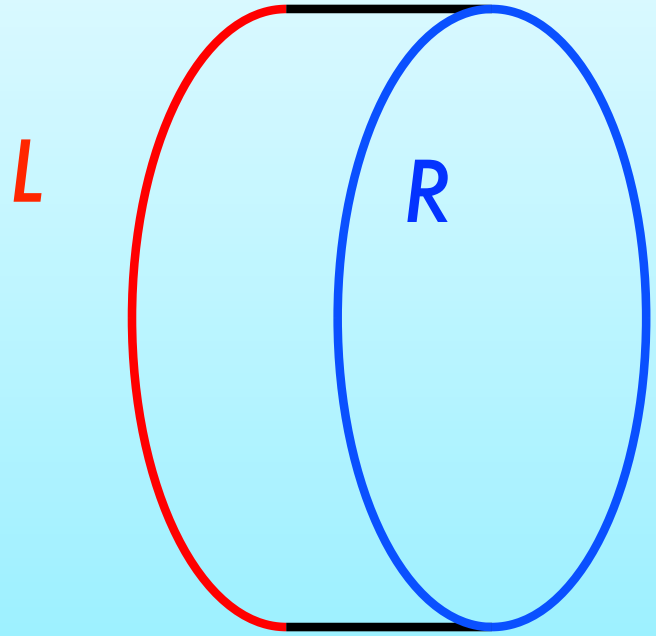
$$\psi_b(r) \sim \frac{1}{\sqrt{a_0^3}} e^{-r/a_0}$$

a_0 :size of the bound state

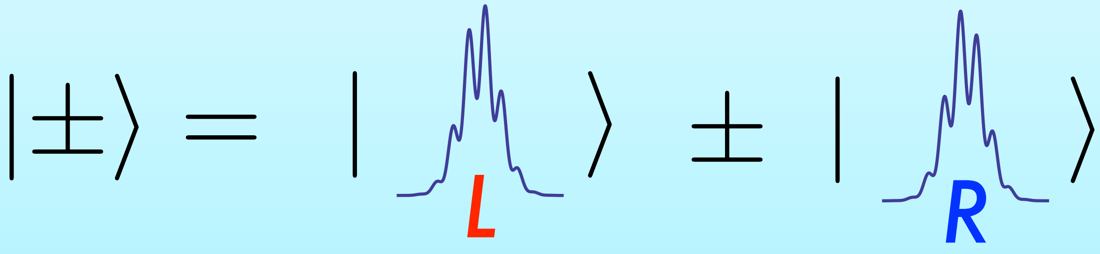
☑ Clear difference only in the infinite system

Right / left to the symmetry

On cylinder



Finite system



☑ Right / left to the symmetry
only in the infinite system,
since the Boundaries are far away !

Bulk-edge correspondence : Emergent principle

c.f. spontaneous symmetry breaking : dynamical

Edge is topological

Why do we care edge states?

Why the Edge States are there??

Accidental ?

NO !

Inevitable reasons

Physical Structures behind:

“Bulk determines the edges”

“Edge determines the bulk”

Bulk-Edge Correspondence

Protected by Topological constraints

Are insulators boring ??

Edge is topological

metal

Gapless excitations

insulator

Excitations need finite energy

☆Metal is useful & interesting

- Successful industrial applications

Classical
Ohm's law up to now

- Also for physicists : Critical, RG,anomalous fermi liquids...

☆Metal is unstable

Peierls instability, Cooper instability ,...

- Exact zero gap excitation needs some protection or fine tuning

Bosonic	Nambu-Goldstone	Breaking of continuos symmetry
Fermionic	Edge states, Domain wall fermions	Topological chiral symmetry Nielsen-Ninomiya
	2D Dirac fermions of 3D TI	Time reversal

- "high energy" effective theory ?

Are insulators boring ??

Edge is topological

metal

Gapless excitations

insulator

Excitations need finite energy

- ★Metal is useful & interesting
- ★Metal is un ... bility ,...
  - Exact zer ... n or fine tuning
  - "high energy" effective theory ?

Lots of new topological phases

- ★Insulator
  - ☆ Band insulators
  - ☆ Superconductors ?!

Insulators are topological !?

Gapped Excitation (mostly)
Stable for perturbation

arch

ed matter course

Topological insulator
as a Quantum Spin Hall state

Edge is topological

"Insulators are stable" implies "Topological"

- Insulators : Gapped
  - Band insulators
  - Superconductors
  - Integer & Fractional Quantum Hall States
  - Valence bond solid (VBS) states
  - Half filled Kondo Lattice
  - Spin Hall insulators
  - Kitaev model & string net

Mostly Stable against interaction!!

Absence of low energy excitations
Energy gap above the ground state

Lots of variety

Absence of fundamental symmetry breaking (mostly)

Quantum/spin liquids (gapped)

Edge is topological

"Insulators are stable" implies "Topological"

Gapped: Nothing in the gap : cf. Nambu-Goldstone boson

No low lying excitations

No Response for small perturbation

~~gapless modes:~~
~~acoustic phonons~~
~~zero sounds~~
~~spin waves~~

??

? ? ?

insulator

Absence of low energy excitations
Energy gap above the ground state

Lots of variety

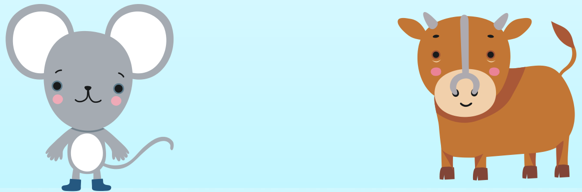
Absence of fundamental symmetry breaking (mostly)

Quantum/spin liquids (gapped)

Edge is topological

"Insulators are stable" implies "Topological"

- Quantum liquids (gapped)
  - Band insulators
  - Superconductors
  - Integer & Fractional Quantum Hall States
  - Integer spin chains (Haldane)
  - Dimer Models (Shastry-Sutherland)
  - Valence bond solid (VBS) states
  - Half filled Kondo Lattice
  - Spin Hall insulators
  - Kitaev model & string net



Topological Order

X.G.Wen '89

Zoo

Something for classification

☐ Topological order

☑ Edge states

☑ Berry connections

Edge is topological

How to understand gapped quantum liquids ?

Lessons from history : Quantum Hall states and Spin 1 chains

QHE

Y. Hatsugai, Phys. Rev. Lett. 71, 3697 (1993)

Bulk-Edge correspondence

Common property of topological ordered states

Bulk

classically featureless

1-st Chern number for QHE

Thouless-Kohmoto-Nightingale-den Nijs

Niu-Thouless-Wu

YH

Edge

low energy localized modes in the gap

edge states for QHE

Laughlin, Halperin, Wen, YH

☑ Edge states

Edge is topological

How to understand gapped quantum liquids ?

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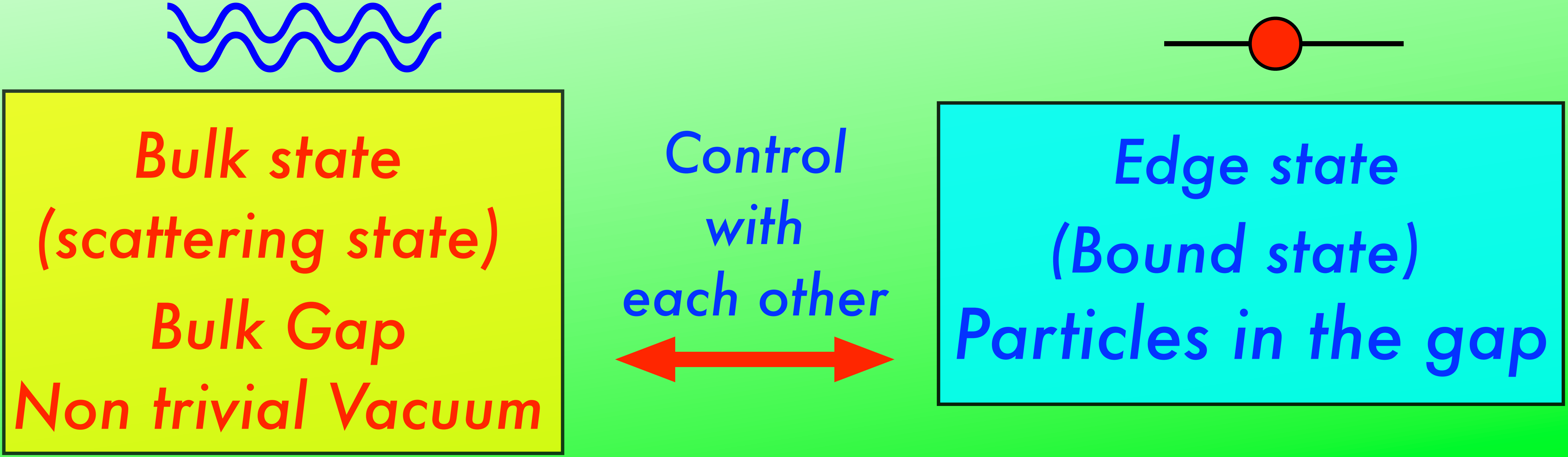
Y. Hatsugai, Phys. Rev. Lett. 71, 3697 (1993)

Bulk-Edge correspondence

Universality

c.f. holographic principle



Bulk state
(scattering state)
Bulk Gap
Non trivial Vacuum

Control
with
each other

Edge state
(Bound state)
Particles in the gap

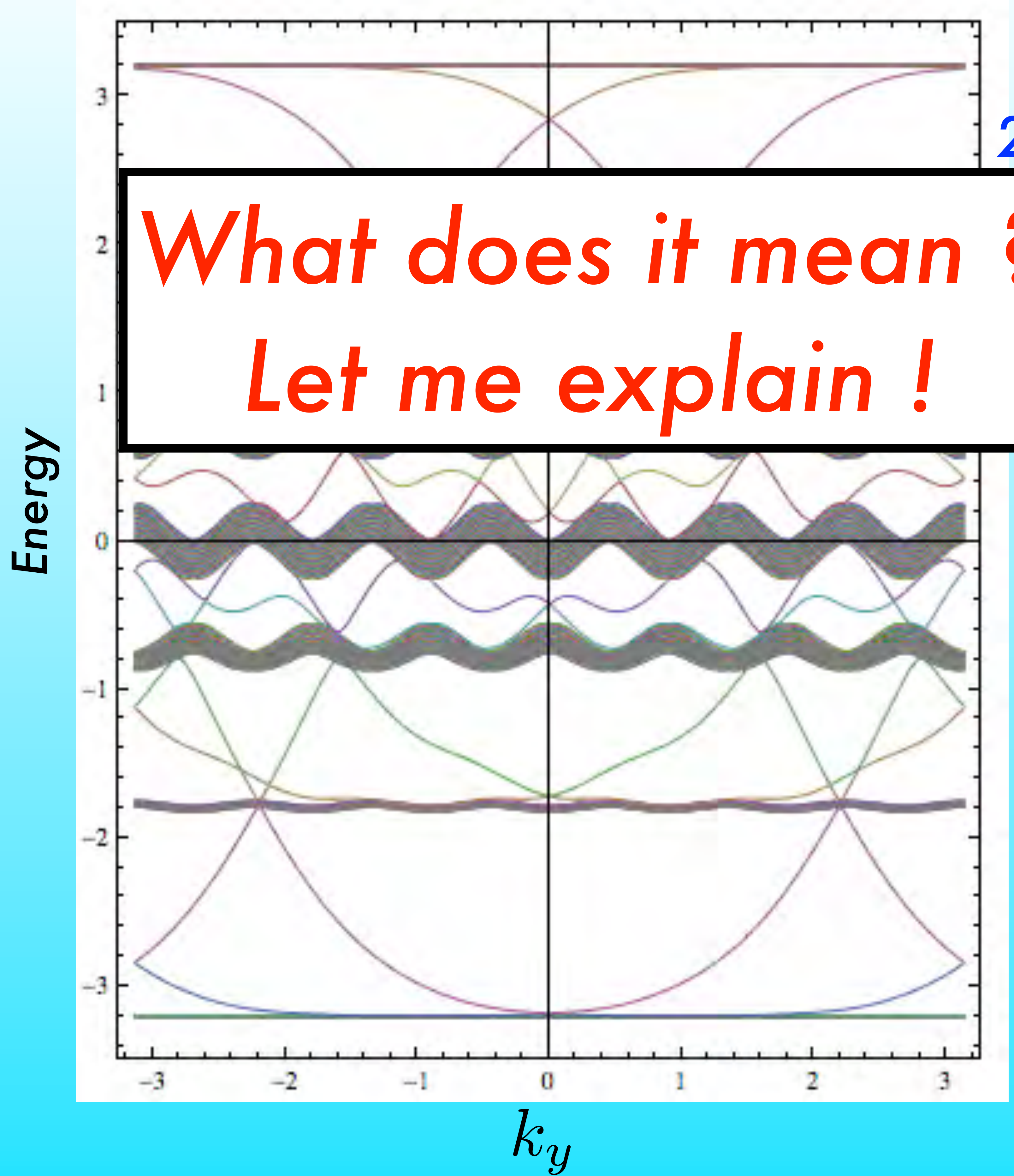
QHE, Spin chains, Graphene, QSHE, ...

☑ Edge states

Hagiwara-Katsumata-Affleck-Halperin-Renard

Edge states of the topological insulators

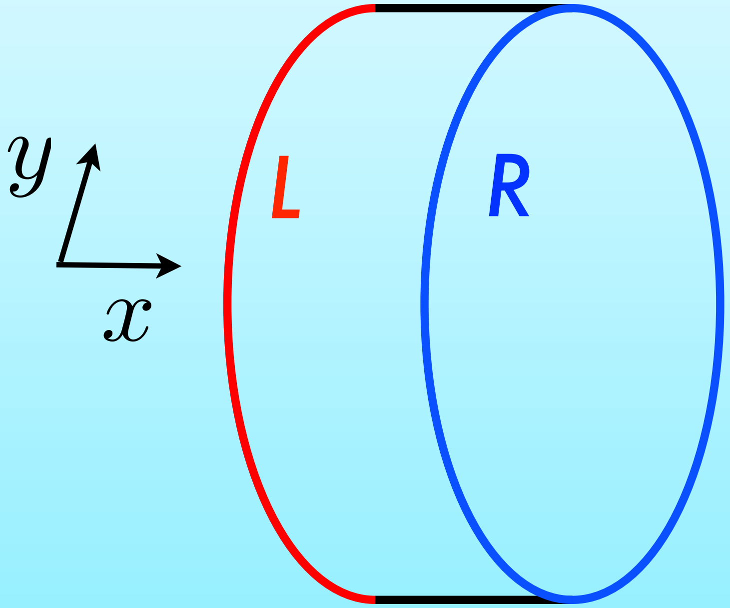
Edge is topological

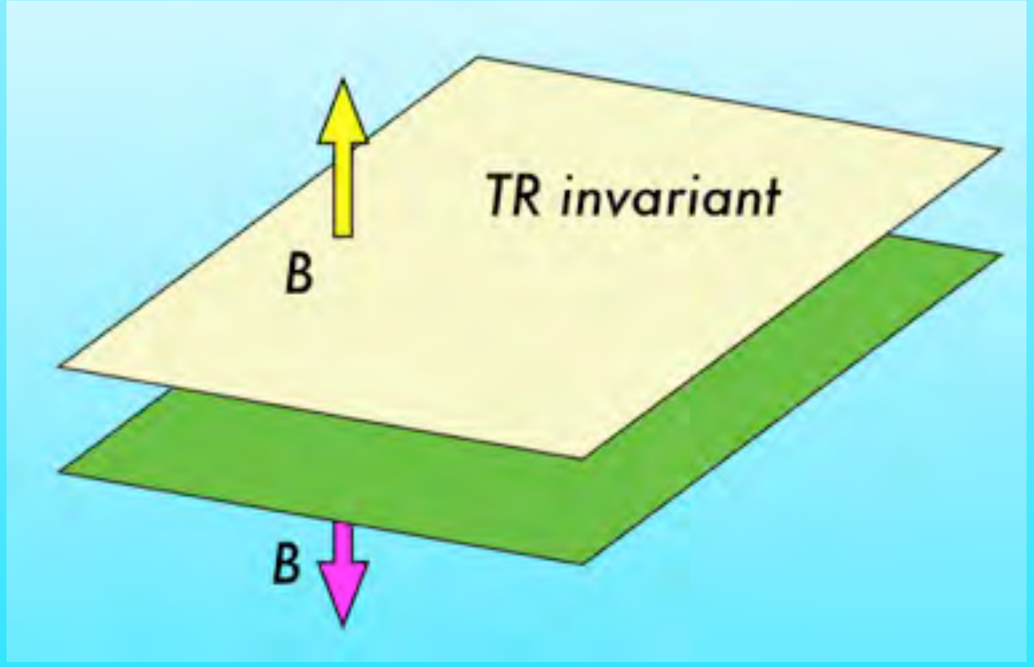


What does it mean ?

Let me explain !

2D QSH state on a cylinder



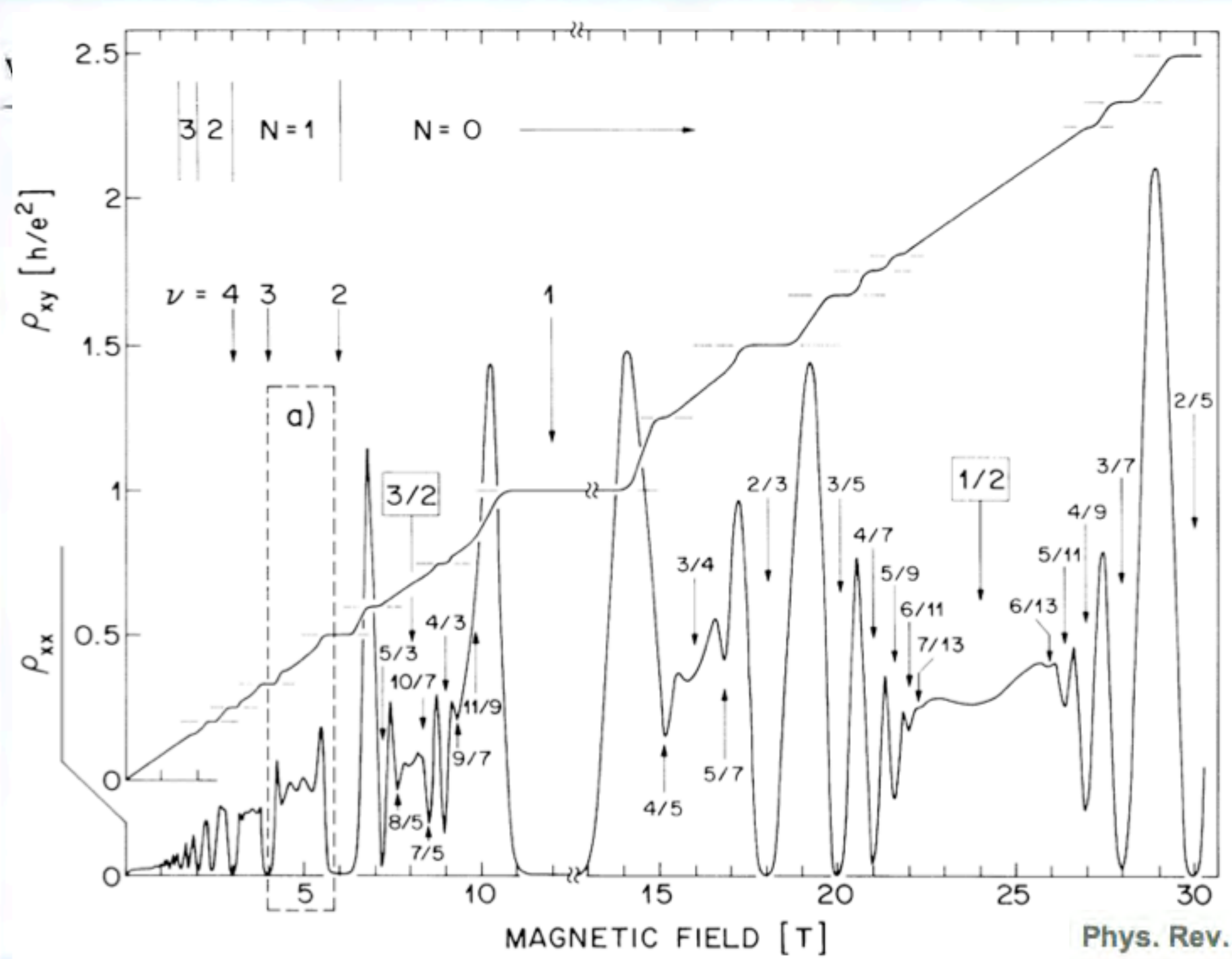


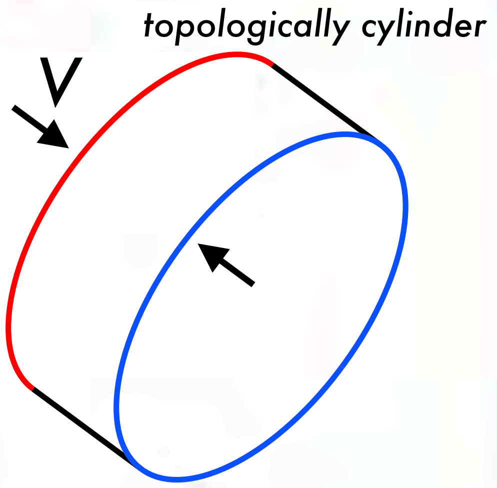
- Topological characterization by edge states
  - Quantization of Hall conductance (graphene as an example)
  - Laughlin argument & edge states
  - Topological number & edge states

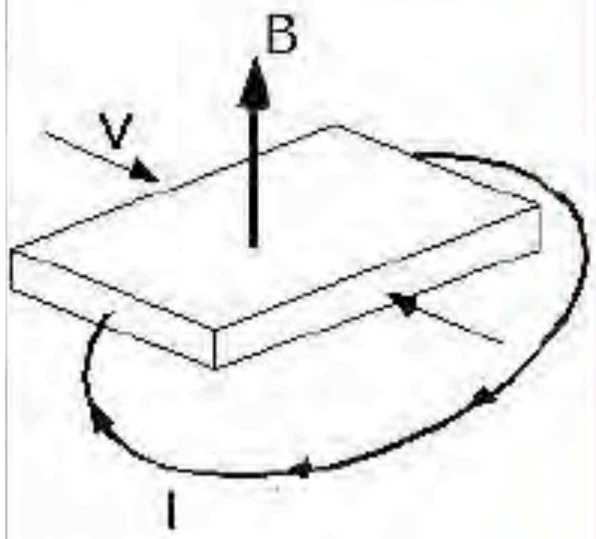
Topol. char. by edges

Quantum Hall Effect *'80, K.v.Klitzing et al.*

Quantization of the Hall conductance σ_{xy} with anomalous accuracy: $I = \sigma_{xy} V$







Phys. Rev. Lett. 59, 1776–1779 (1987)

R. Willett et al.

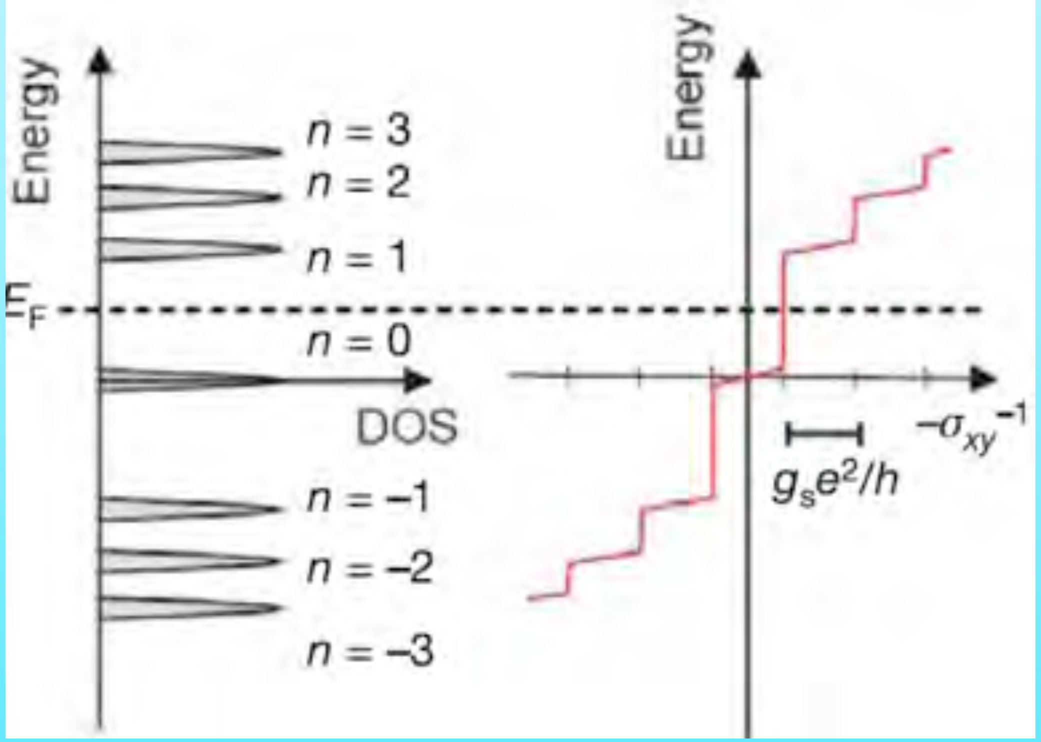
Graphene

Topol. char. by edges

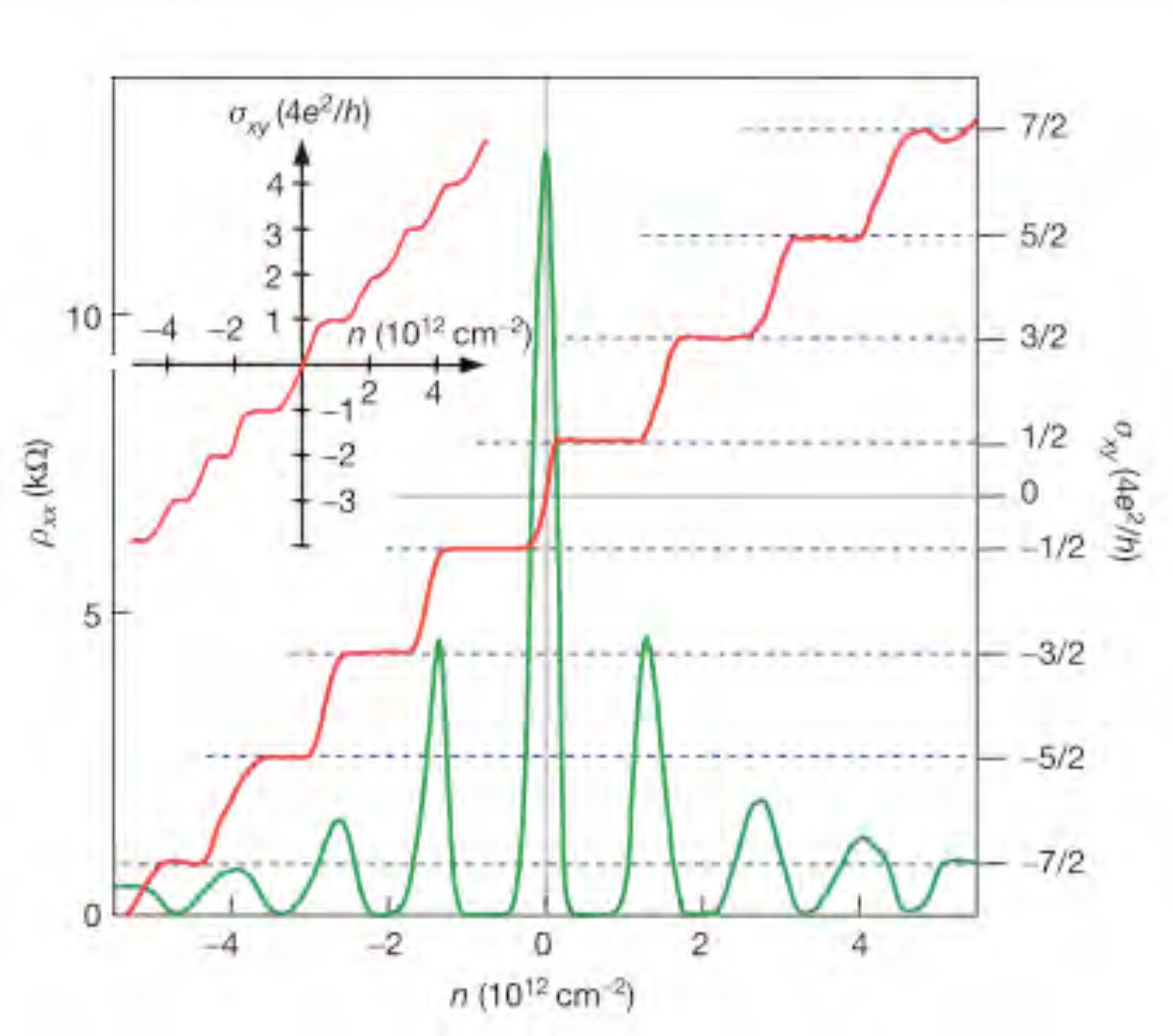
☆ Anomalous QHE of gapless Dirac Fermions

$$\sigma_{xy} = \frac{e^2}{h}(2n+1), \ \ n = 0, \pm 1, \pm 2, \cdots$$

$$= 2\frac{e^2}{h}(n+\frac{1}{2})$$



Zhang et al. Nature 2005

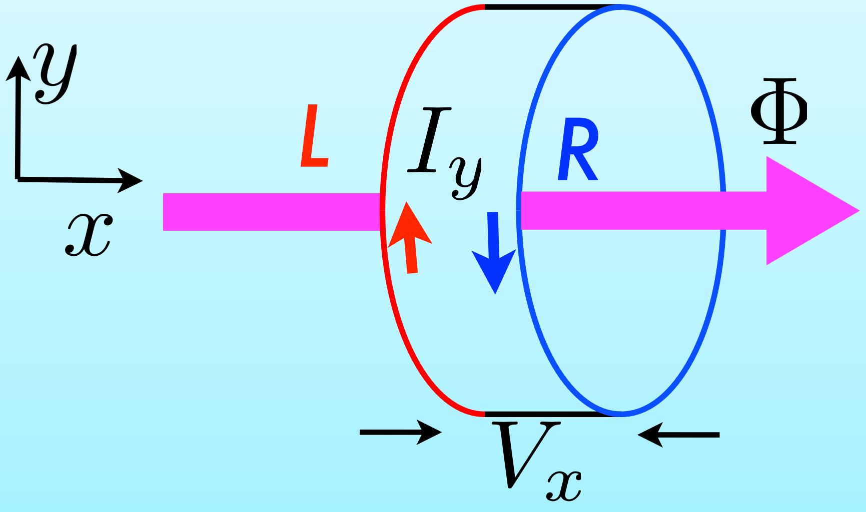


Novoselov et al. Nature 2005

Stability of the quantized Hall Conductance

☆ Gauge invariance and quantization of σ_{xy} Laughlin '81

adiabatic process to increase Φ



$$A \to A' = A + \nabla\Phi, \quad \delta\Phi = \int_{\bigcirc}(A' - A)$$

$$\psi \to \psi' = e^{i2\pi\delta\Phi/\Phi_0}\,\psi$$ one particle state

$$\delta\Phi = \Phi_0 = \frac{e}{h} \longrightarrow \psi' = \psi$$

flux quantum

Gauge transformation

Byers-Yang formula

$$I_y = \frac{\Delta E}{\Delta\Phi} = \frac{neV_x}{h/e} = \overset{\sigma_{yx}}{\boxed{n(\frac{e^2}{h})}}V_x = \sigma_{yx}V_x$$

$$\Delta\Phi = \Phi_0 = \frac{h}{e}, \quad \Delta E = n \cdot eV_x$$

All states are invariant up to phase after the process:

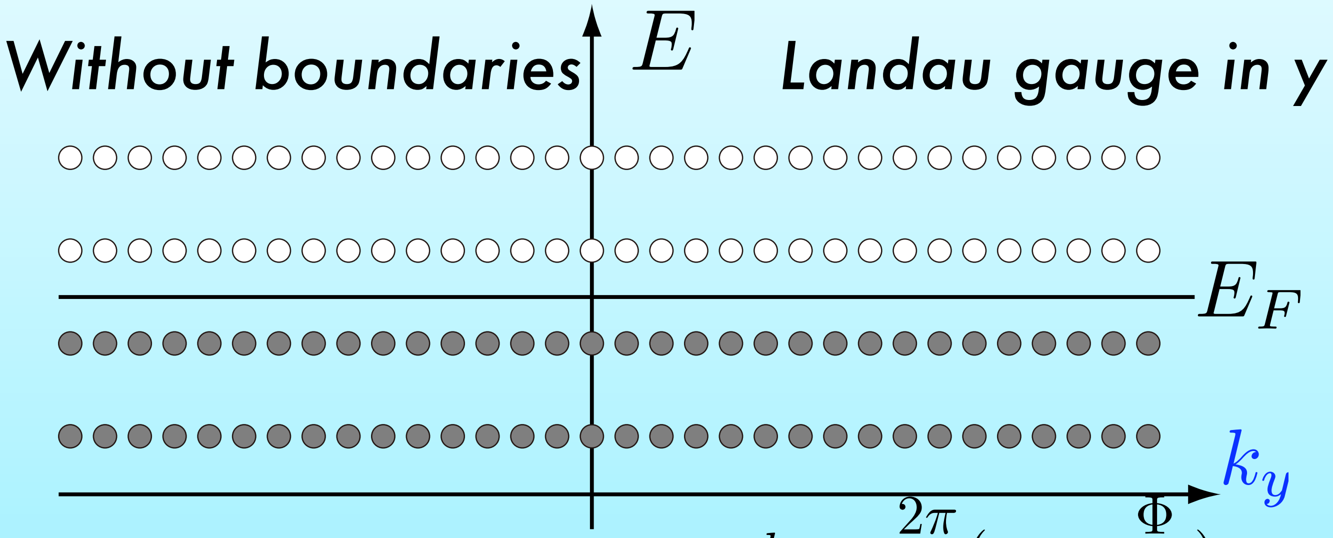
Some n states are carried from L to the R

n : generic integer (but undetermined)

Stability of the quantized Hall Conductance

☆ Edge states and Hall conductance σ_{xy} Halperin '82

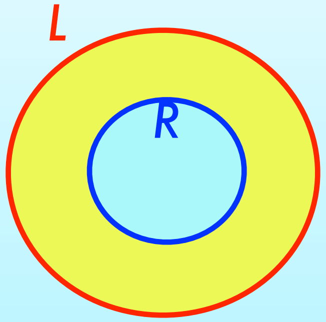
Without boundaries



$$H_{2D} = \sum_{k_y} H_{1D}(k_y)$$

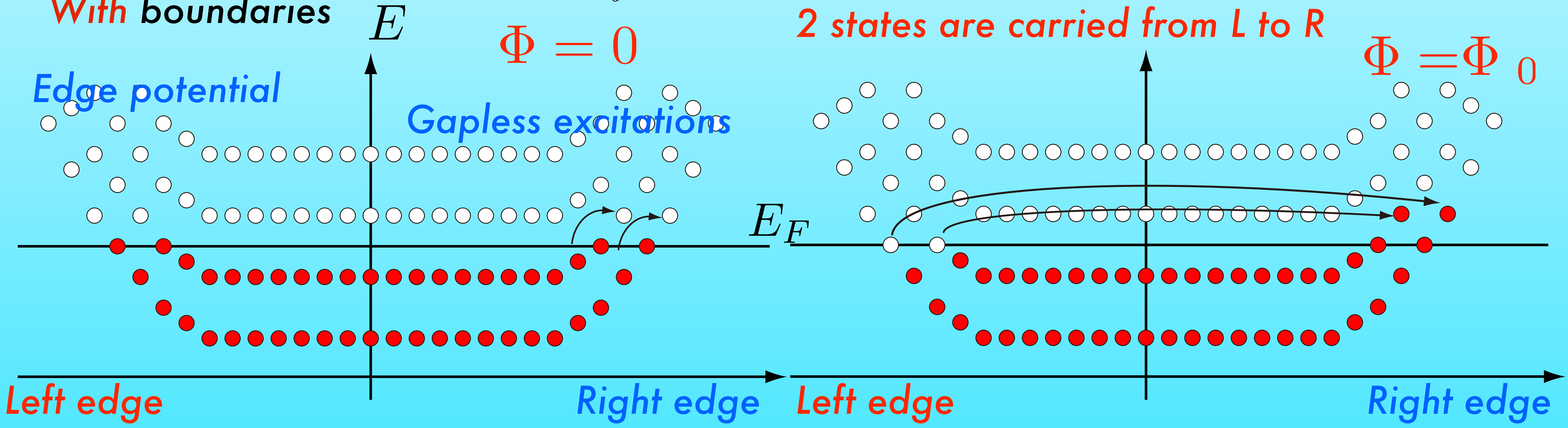
$H_{1D}(k_y)$: harmonic osc. centered at

$$\langle x \rangle \sim \ell_B^2 k_y$$



$$k_y = \frac{2\pi}{L_y}(n_y + \frac{\Phi}{\Phi_0}) \quad n_y = 0, \pm 1, \pm 2, \cdots$$

With boundaries



Laughlin's undetermined n : # of Landau Levels below E_F

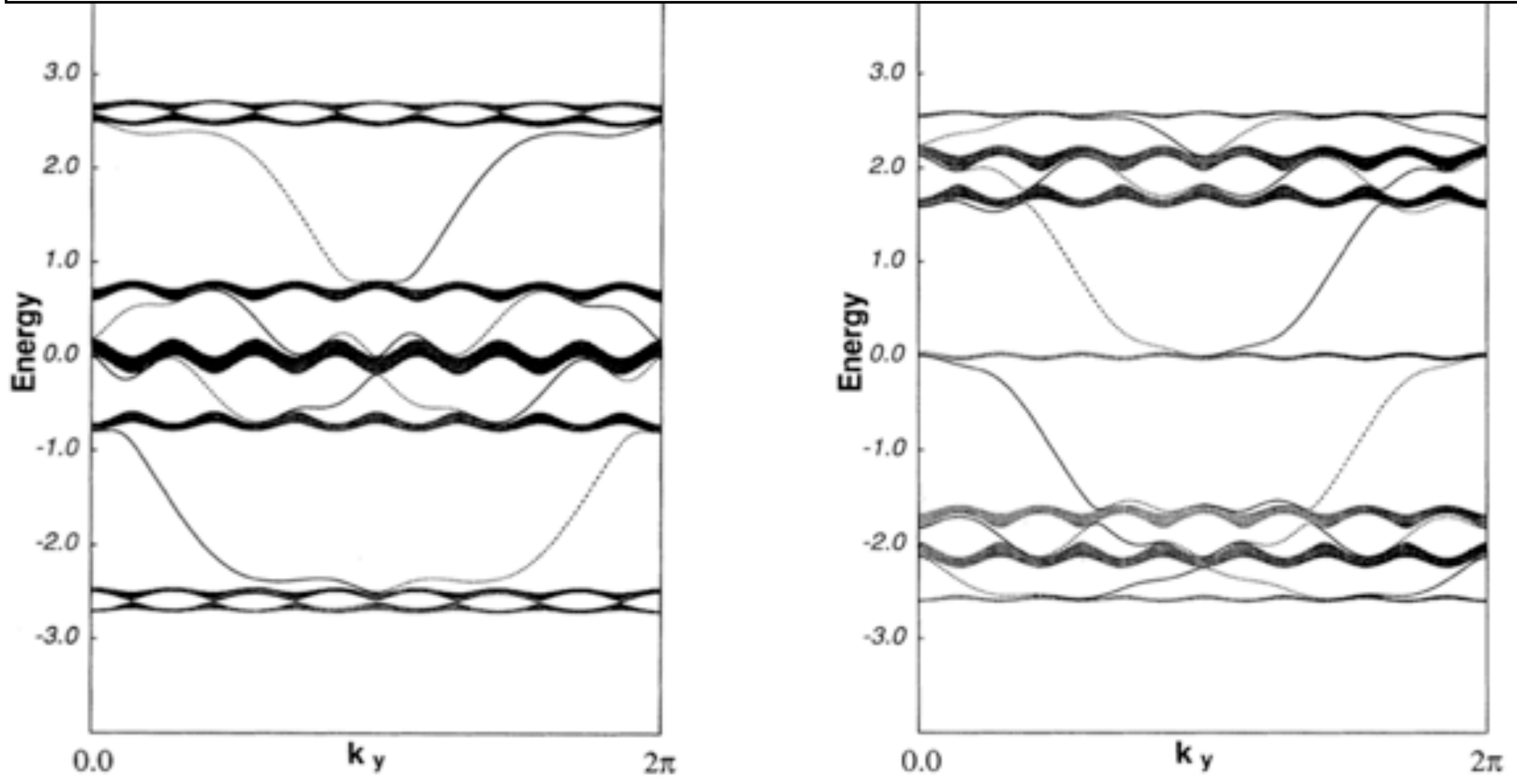
Edge states are essential in the QHE !

Quantum Hall edge states : Lattice electrons on cylinder

Topol. char. by edges

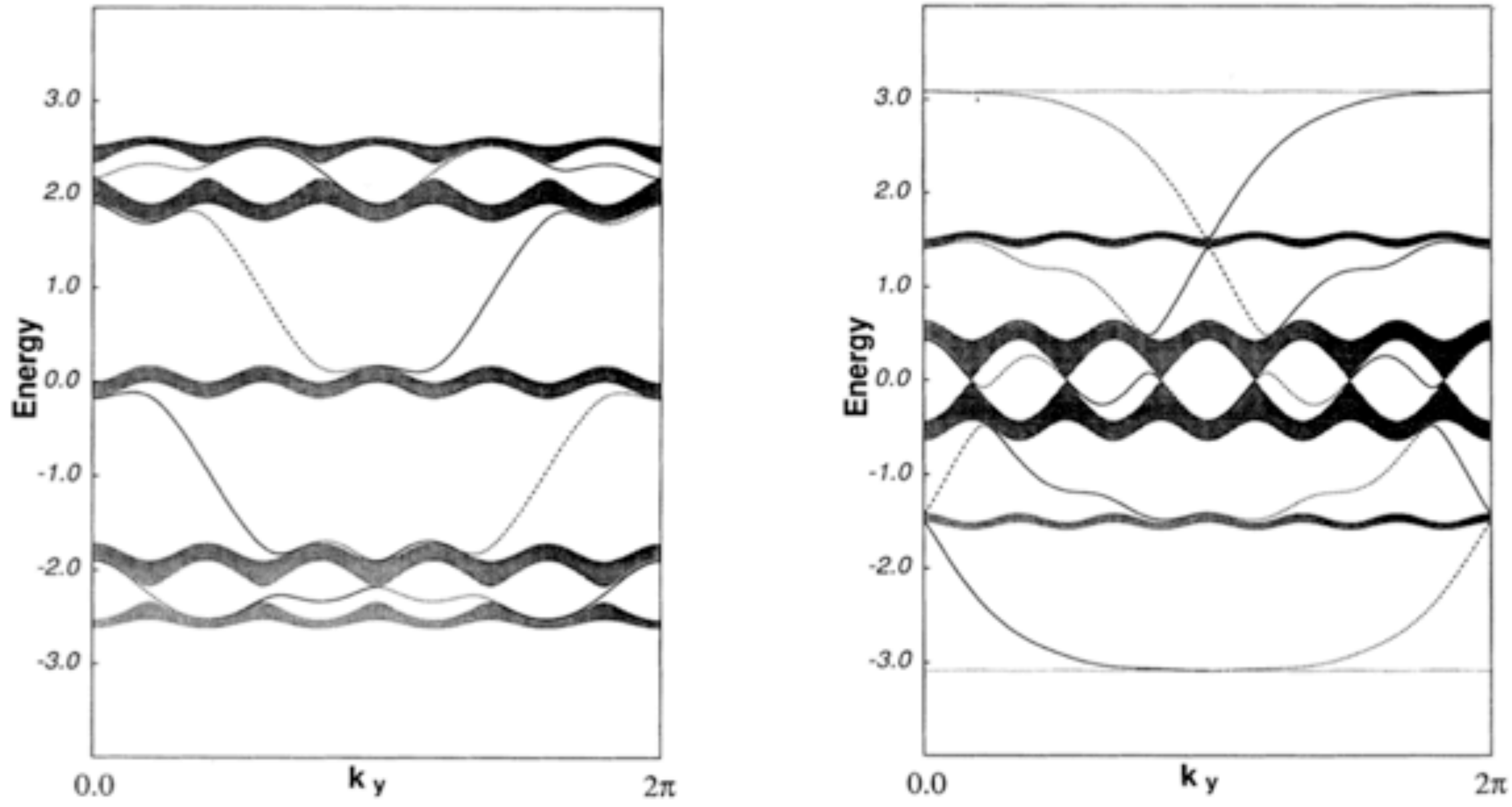
11 8

In gap states as the edge states

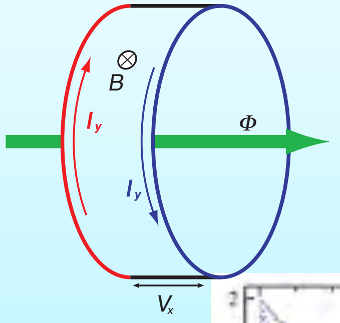


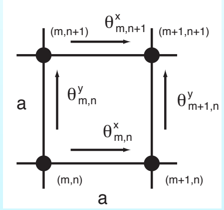
(c) p= 2,q= 5 ; r=1.0

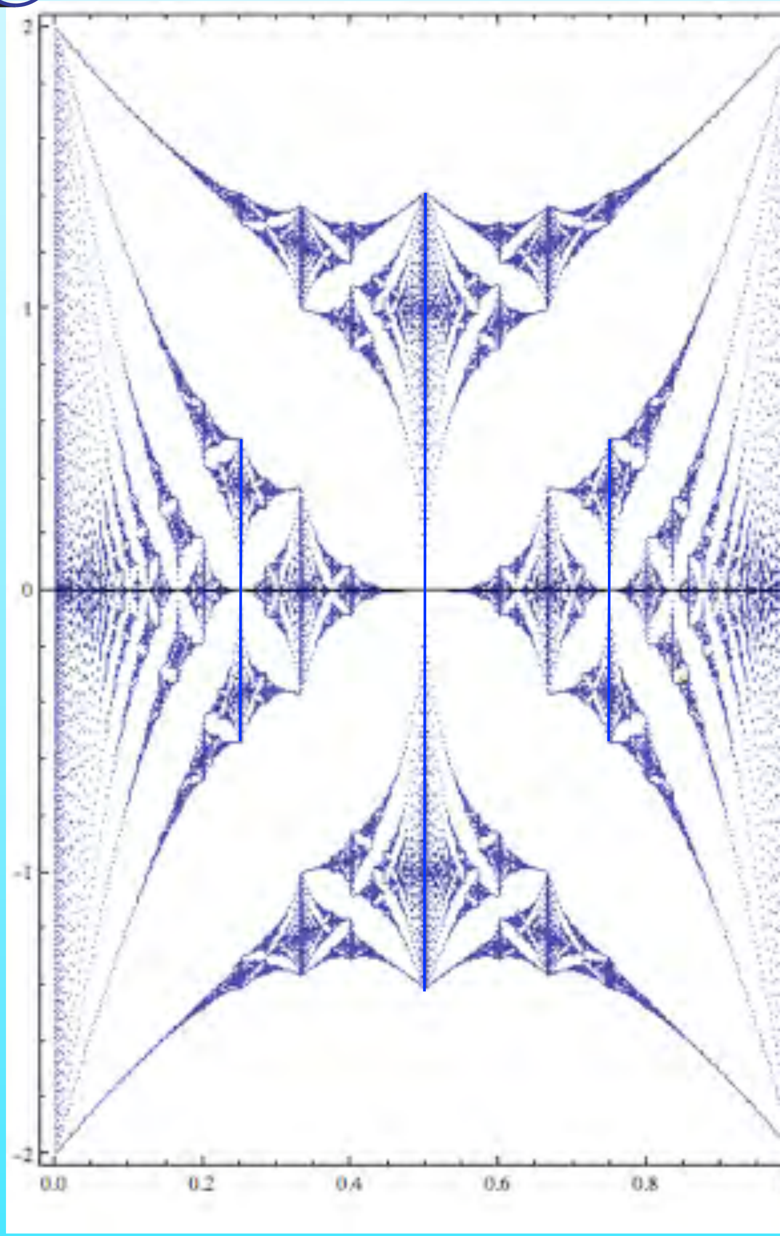
(d) p= 1,q= 6 ; r=1.0



Y. H, Phys. Rev. B 48, 11851–11862 (1993)







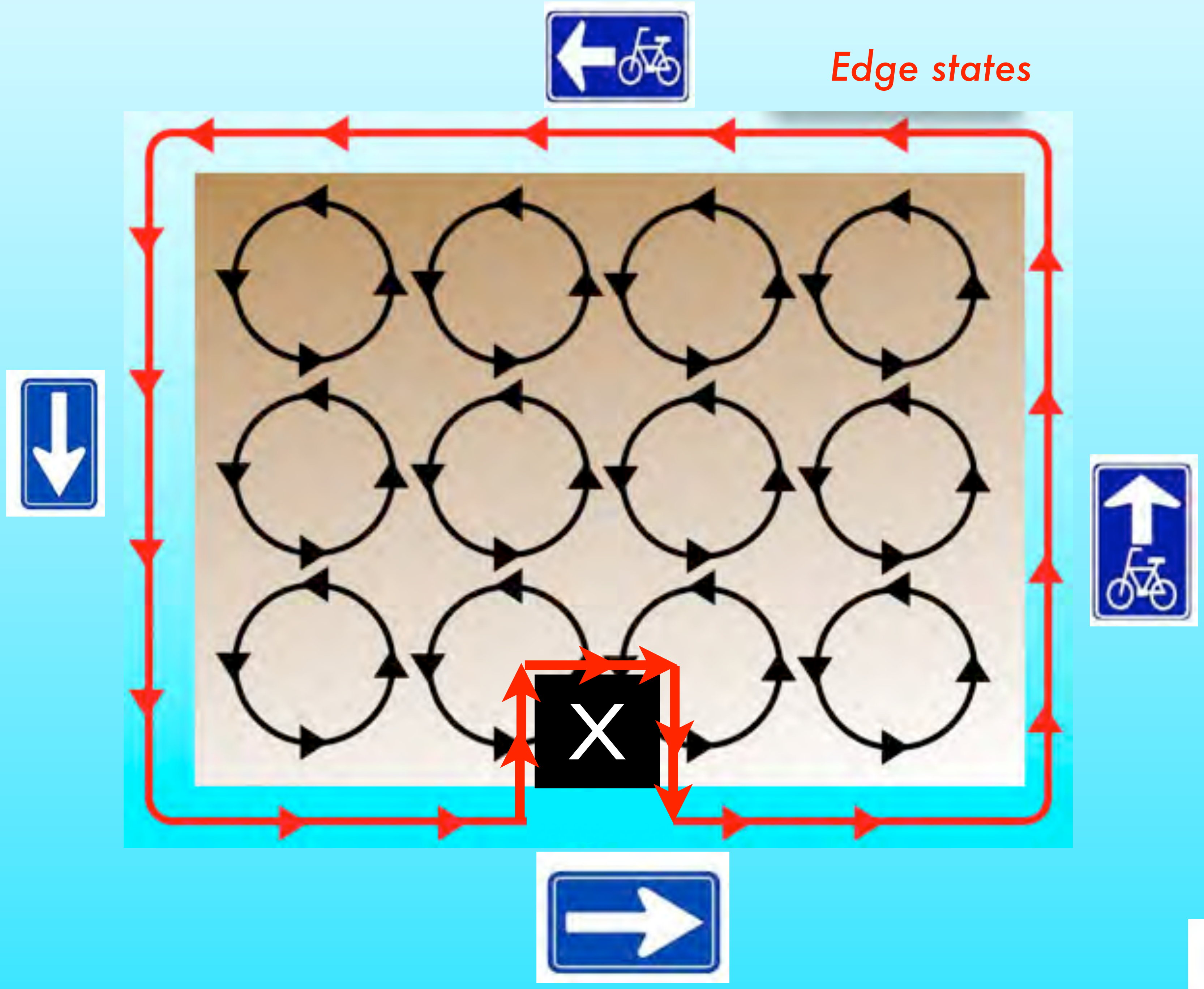
C. Albrecht, J.H. Smet, K. von Klitzing, D. Weiss, V. Umansky, H. Schweizer: Evidence of Hofstadter's Fractal Energy Spectrum in the Quantized Hall Conductance. Phys. Rev. Lett. 86(1), 147-150 (2001).

Experimentally realized

Edge states are topologically stable

Cyclotron motion by Lorentz force $\boldsymbol{F} = -e\boldsymbol{v} \times \boldsymbol{B}$

Currents are canceled in the bulk but induces a boundary current



Edge states are chiral

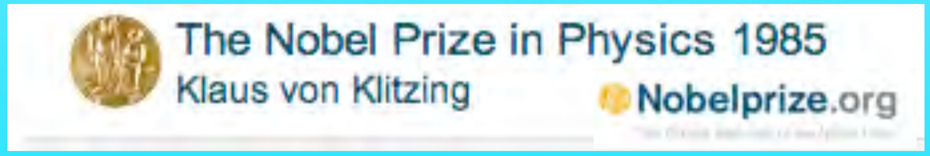
One way going !!

Cannot stop !

No back scattering

Stable for impurities !!

Topological stability
of
Chiral edge states


The Nobel Prize in Physics 1985
Klaus von Klitzing
Nobelprize.org

Edge States of Graphene

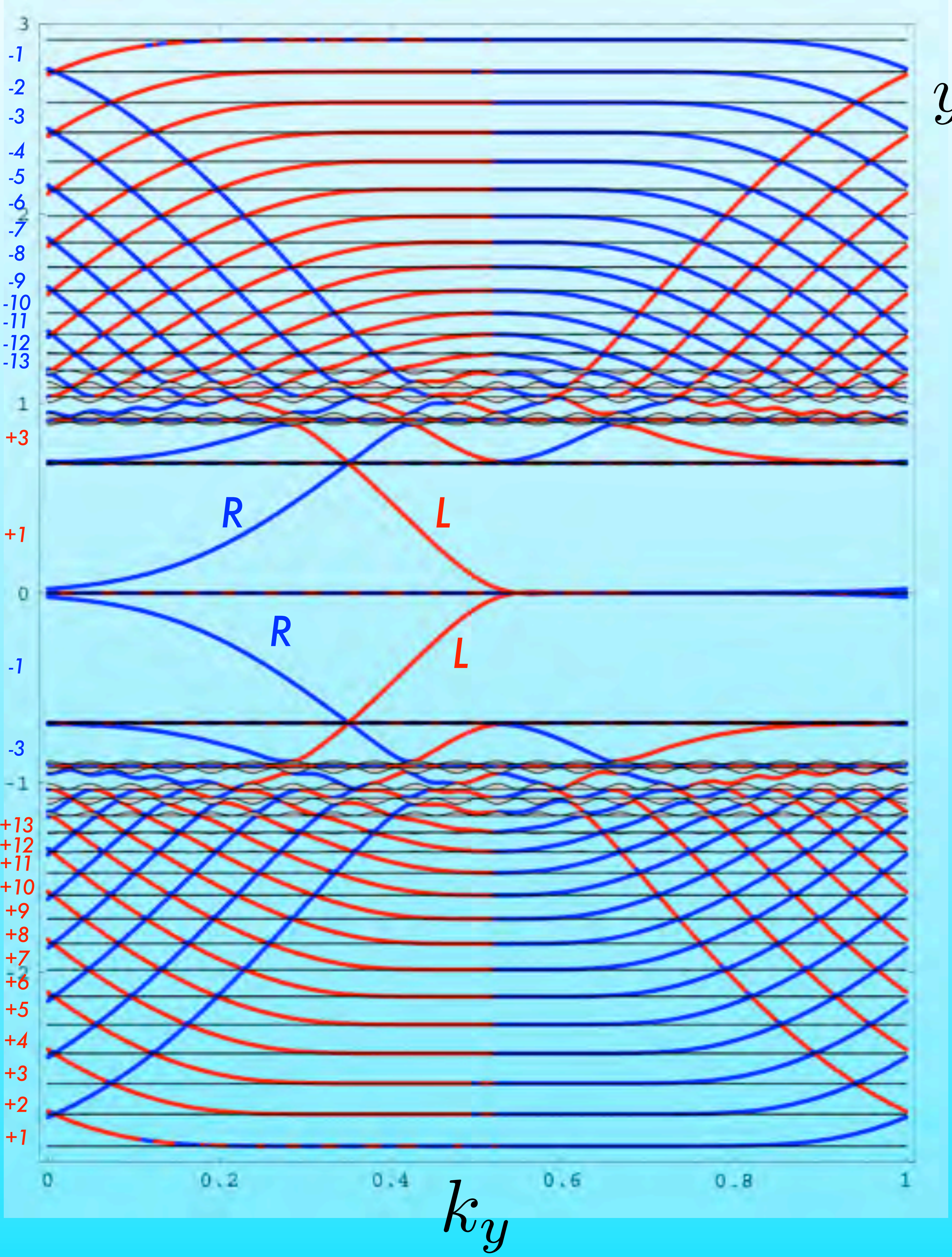
Topol. char. by edges

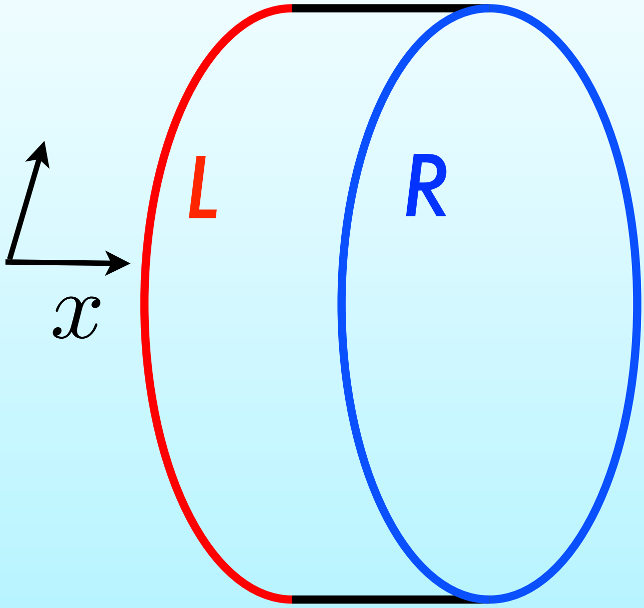
Standard Quantization (hole)

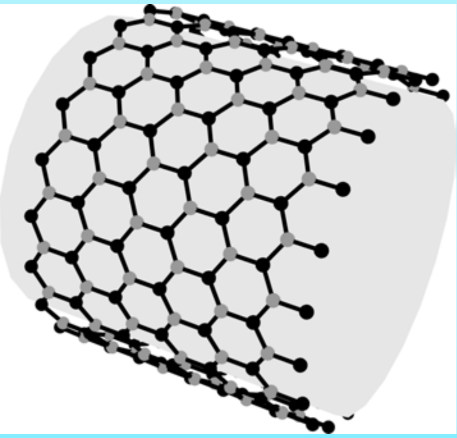
$\phi = 1/21$

Dirac Type Quantization

Standard Quantization



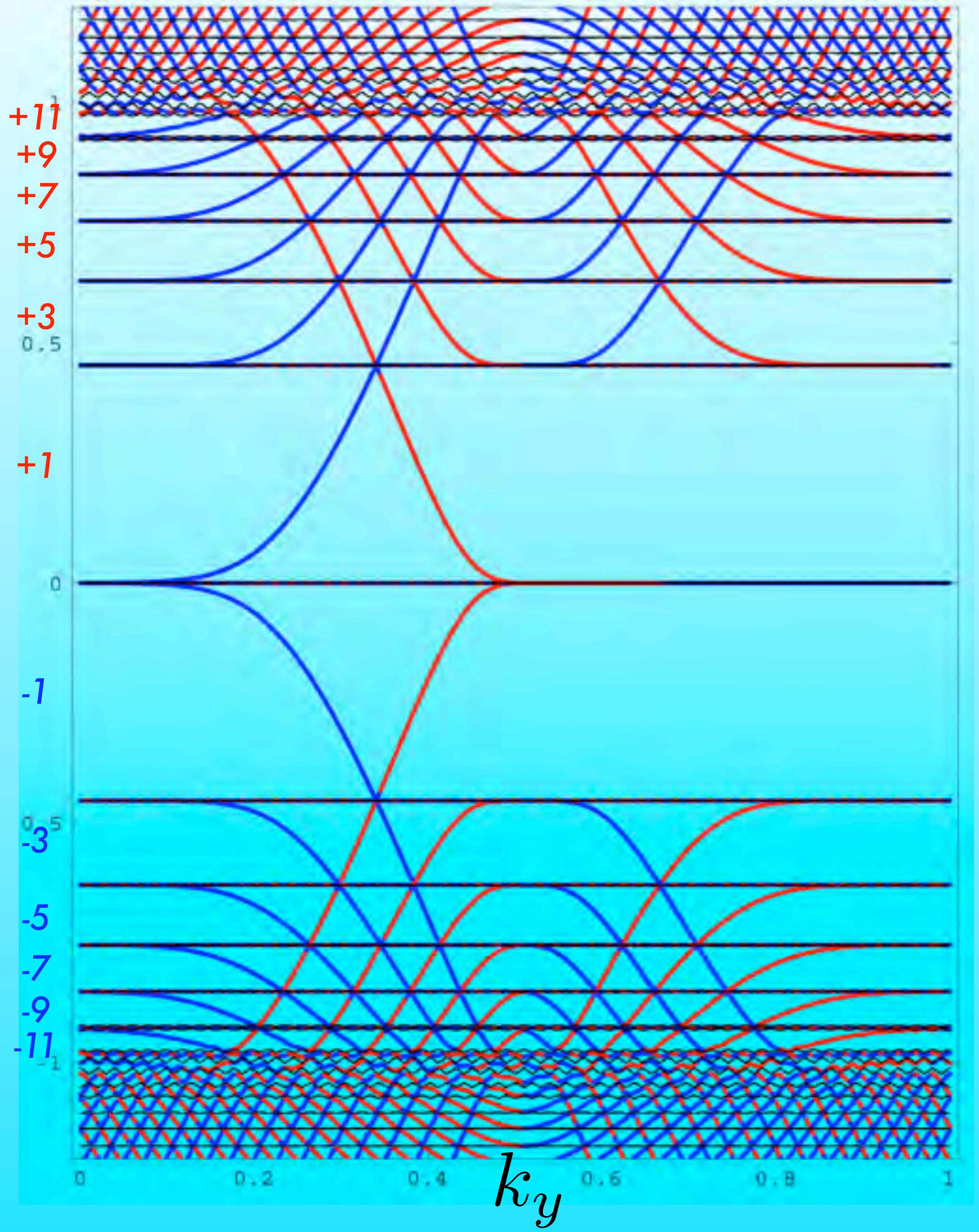




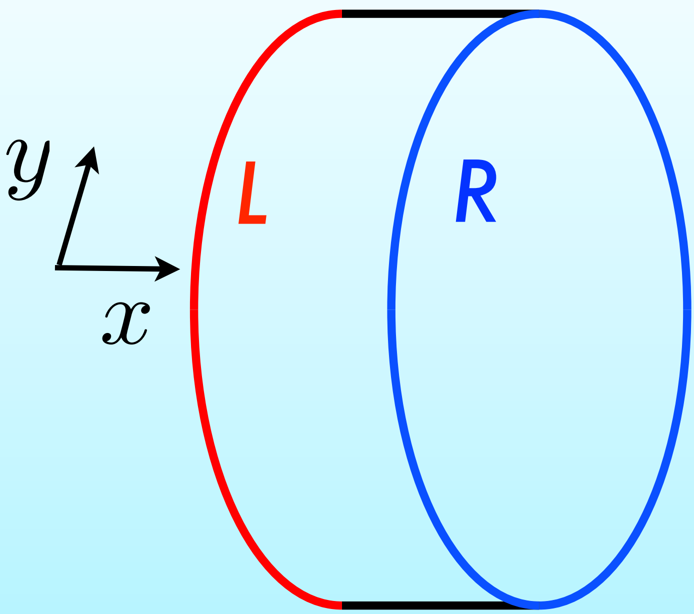
Edge States of Graphene

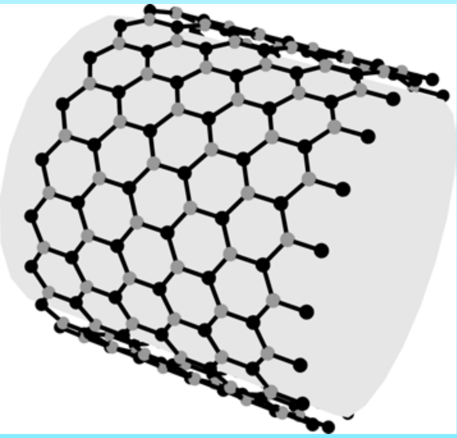
Topol. char. by edges

$\phi = 1/51$



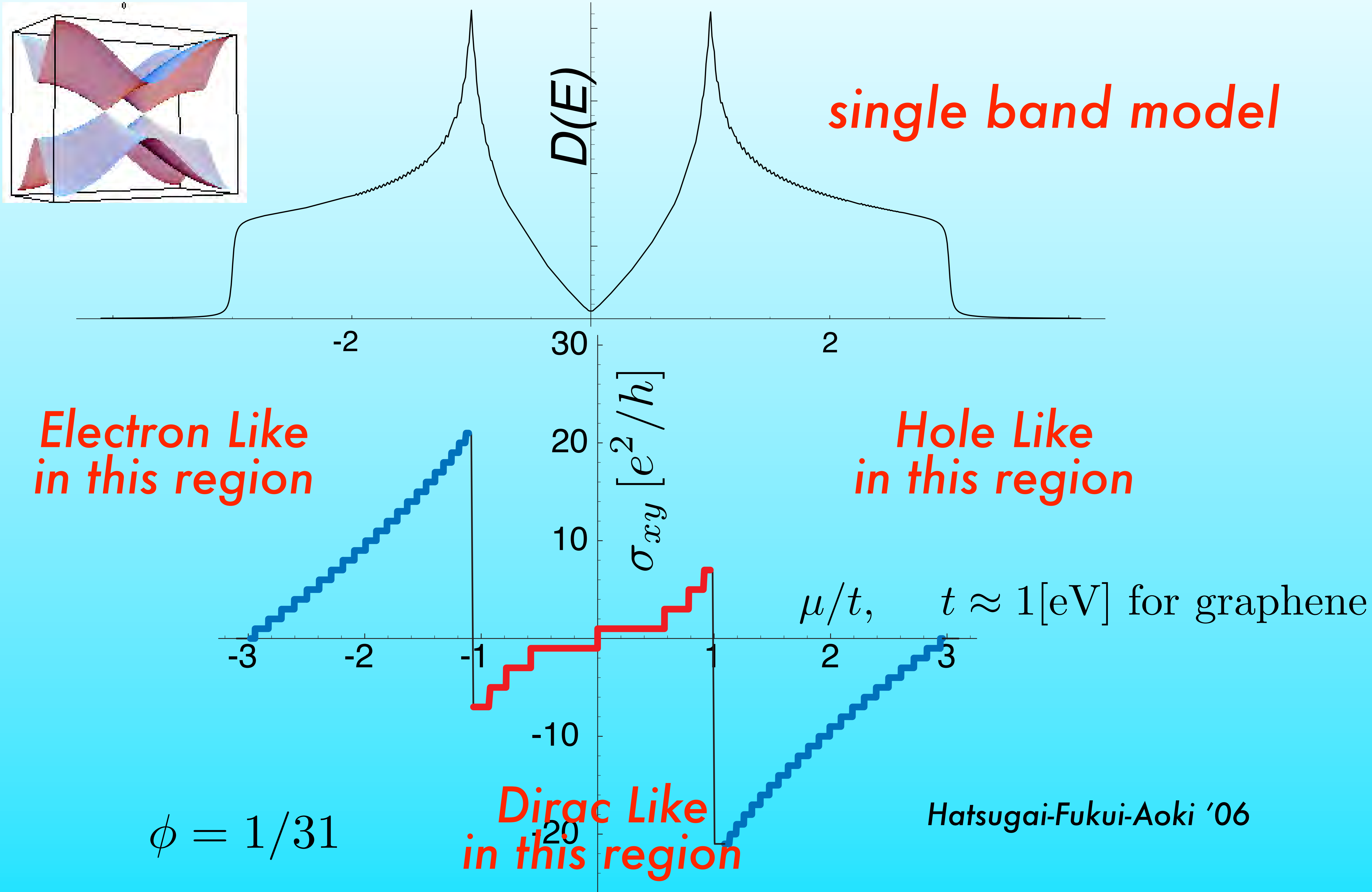
Edge States being consistent with Dirac Type Quantization





Hall Conductace vs chemical potential

- Accurate Hall conductance over whole spectrum



single band model

Electron Like
in this region

Hole Like
in this region

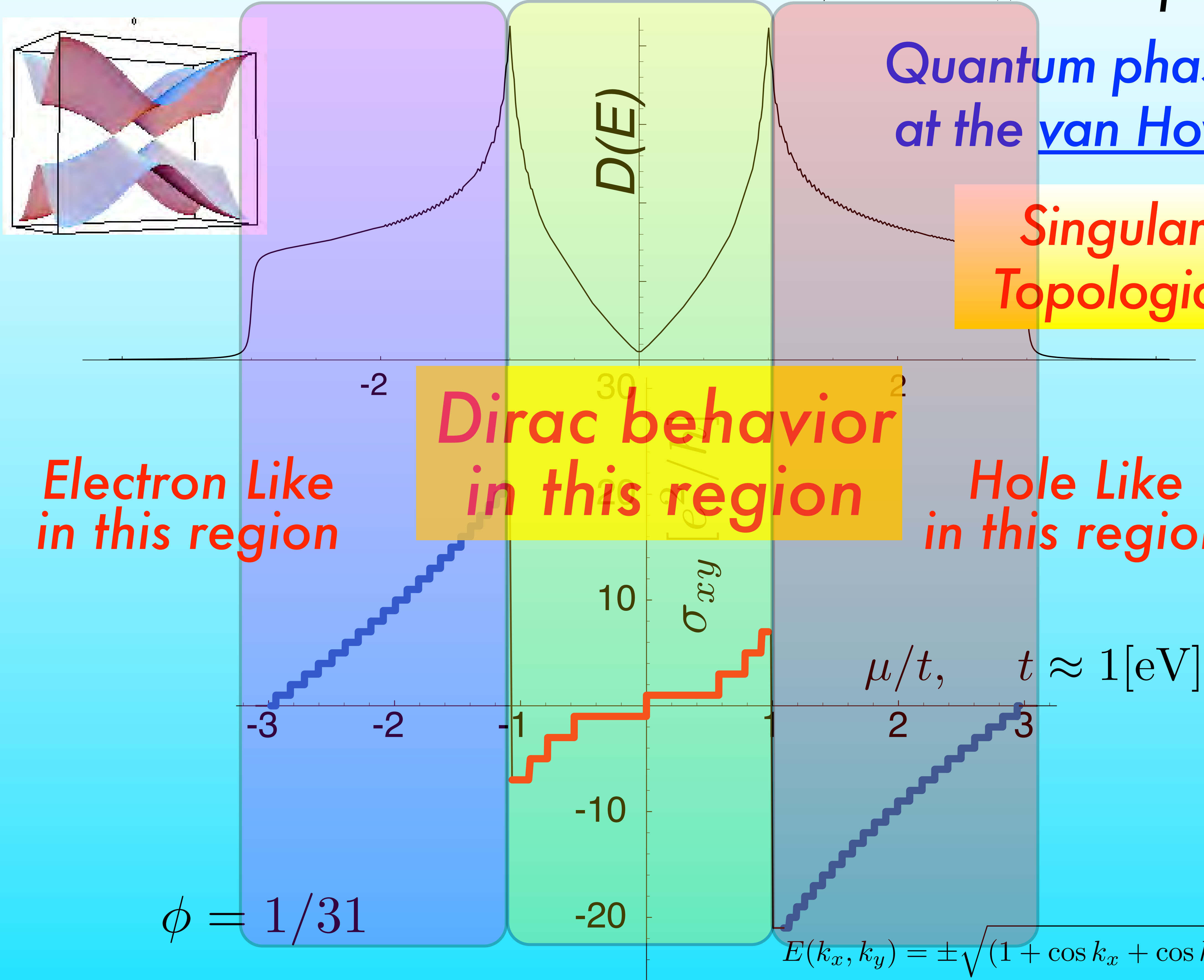
Dirac Like
in this region

$\phi = 1/31$

Hatsugai-Fukui-Aoki '06

Hall Conductace vs chemical potential

- ☆ Accurate Hall conductance over the whole spectrum



Quantum phase transition
at the van Hove Energies

Singularity breaks
Topological Stability

Electron Like
in this region

Dirac behavior
in this region

Hole Like
in this region

$\mu/t, \quad t \approx 1[\mathrm{eV}]$ for graphene

$\phi = 1/31$

$$E(k_x, k_y) = \pm\sqrt{(1+\cos k_x + \cos k_y)^2 + (\sin k_x + \sin k_y)^2}$$

How the edge states determine ?

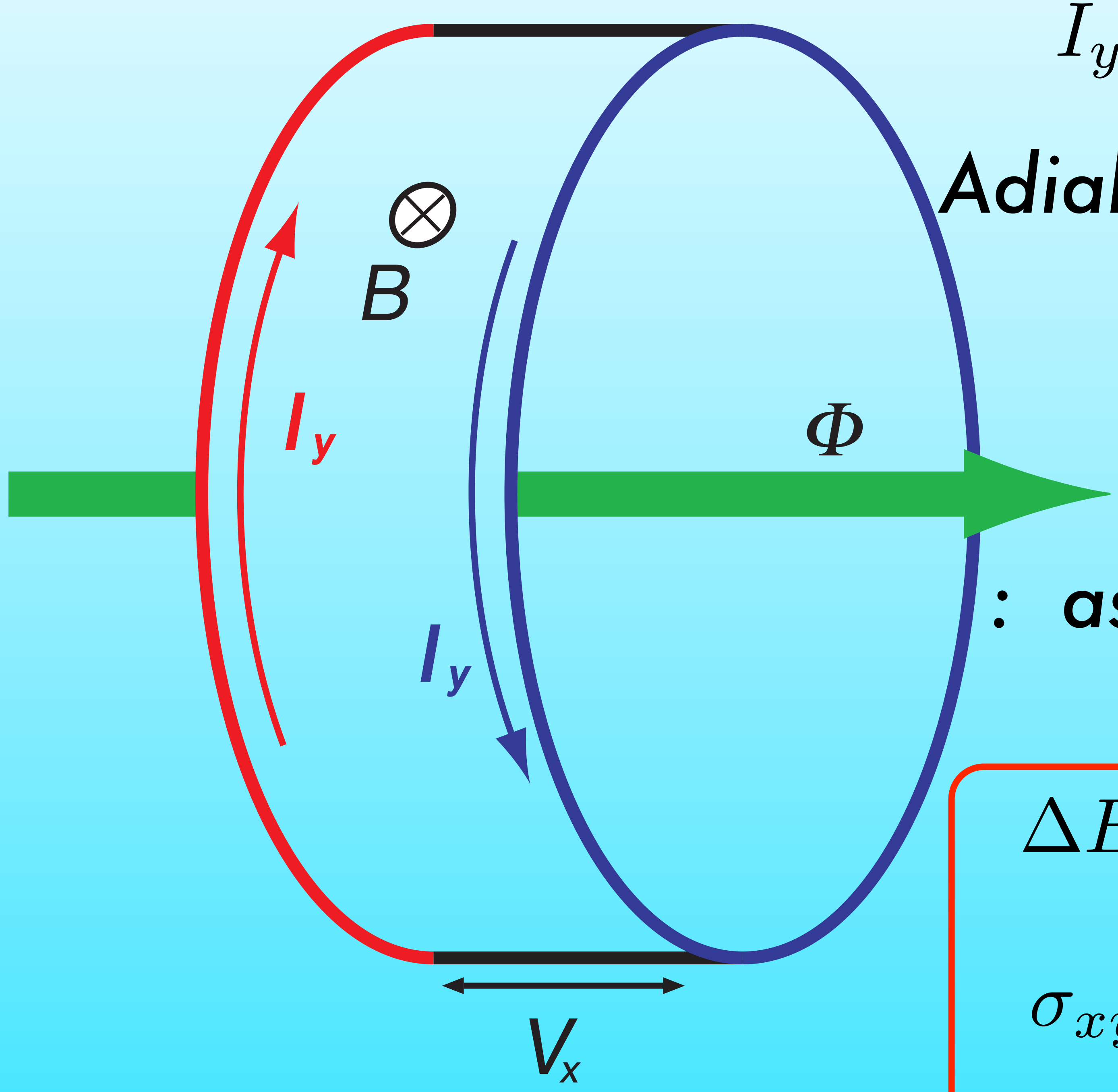
How to calculate σ_{xy} by the edge states?

Topol. char. by edges

Laughlin's Argument & Edge States

☆ Gauge Invariance & Byers-Yang' Formula

Laughlin '81



$$I_y = \frac{\Delta E}{\Delta \Phi} = \sigma_{xy} V_x$$ Byers-Yang

Adiabatic increase by $\Delta\Phi = \Phi_0 = \frac{h}{e}$

→ Insulating System goes back to the Original State

: assume n electrons are carried from the left to the right

$$\Delta E = n e V_x$$

$$\sigma_{xy} = \frac{e^2}{h} n$$

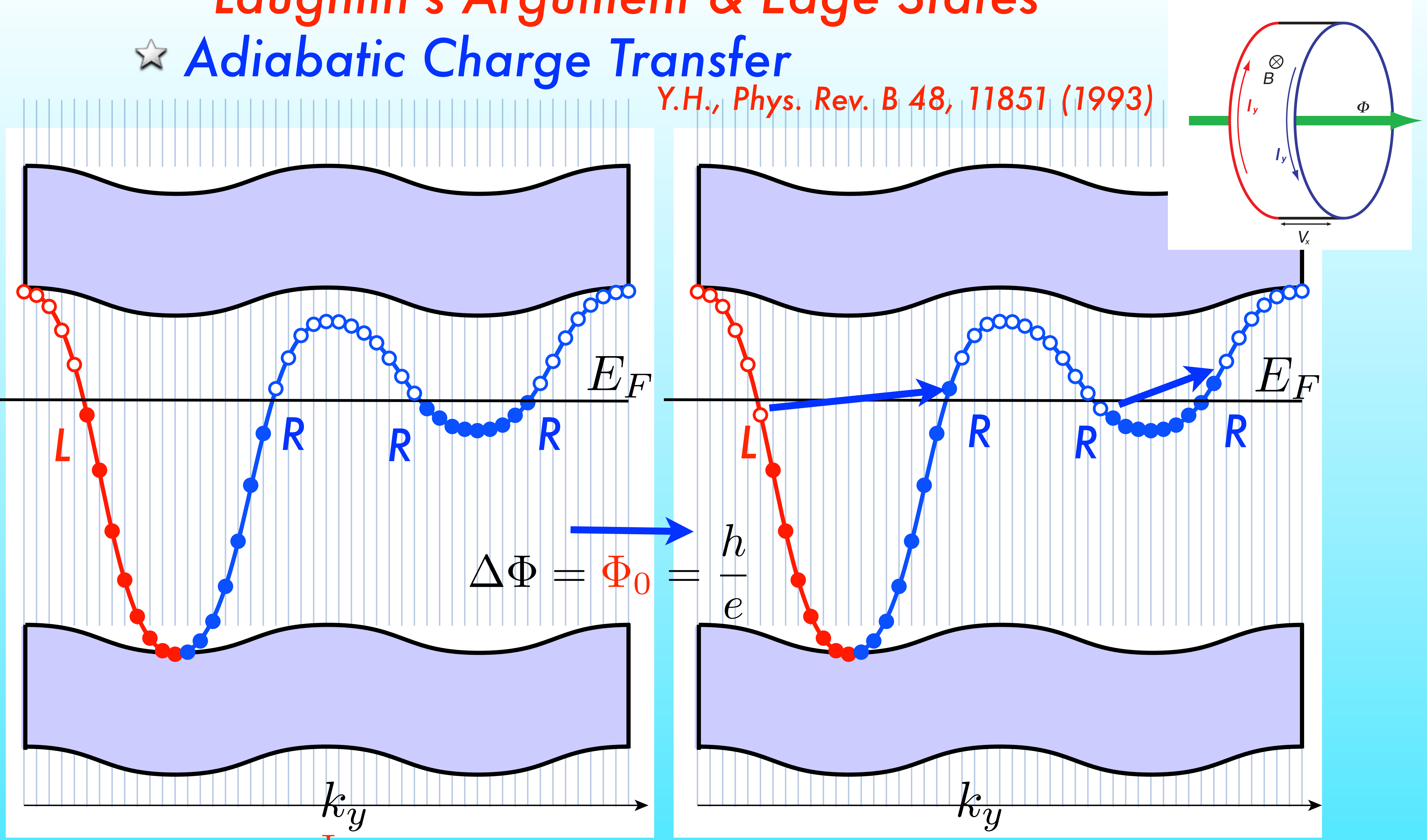
n is an integer
but
unknown

Quantization of σ_{xy} by Edge states

Laughlin's Argument & Edge States

☆ Adiabatic Charge Transfer

Y.H., Phys. Rev. B 48, 11851 (1993)



$$\Delta\Phi = \Phi_0 = \frac{h}{e}$$

$$k_y = 2\pi \frac{n + \frac{\Phi}{\Phi_0}}{L_y}, \quad n: \ \text{integers}$$

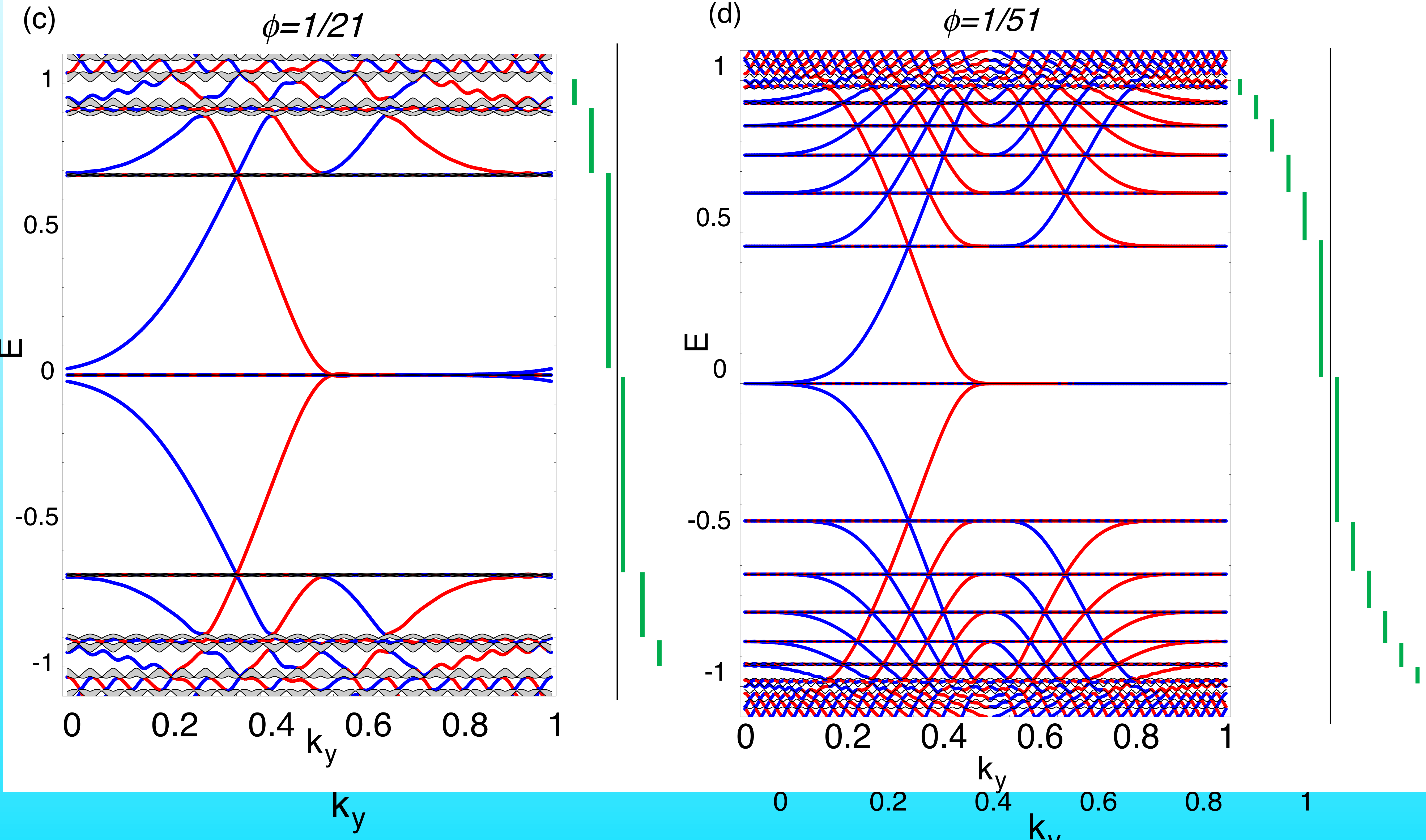
1 Electron is carried from the Left
to the right in this case

$$\sigma_{xy} = \frac{e^2}{h} \cdot 1$$

Bulk – Edge Correspondence ?

☆ *Numerically* $\sigma_{xy}{}^{\mathrm{bulk}} = \sigma_{xy}{}^{\mathrm{edge}}$

Near Zero

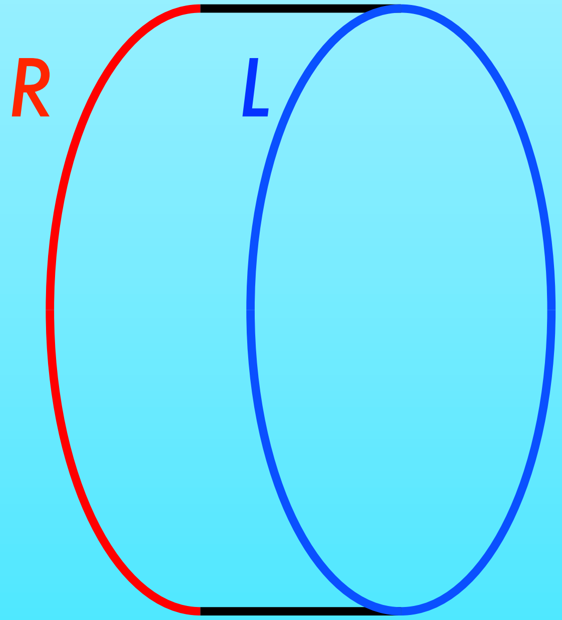


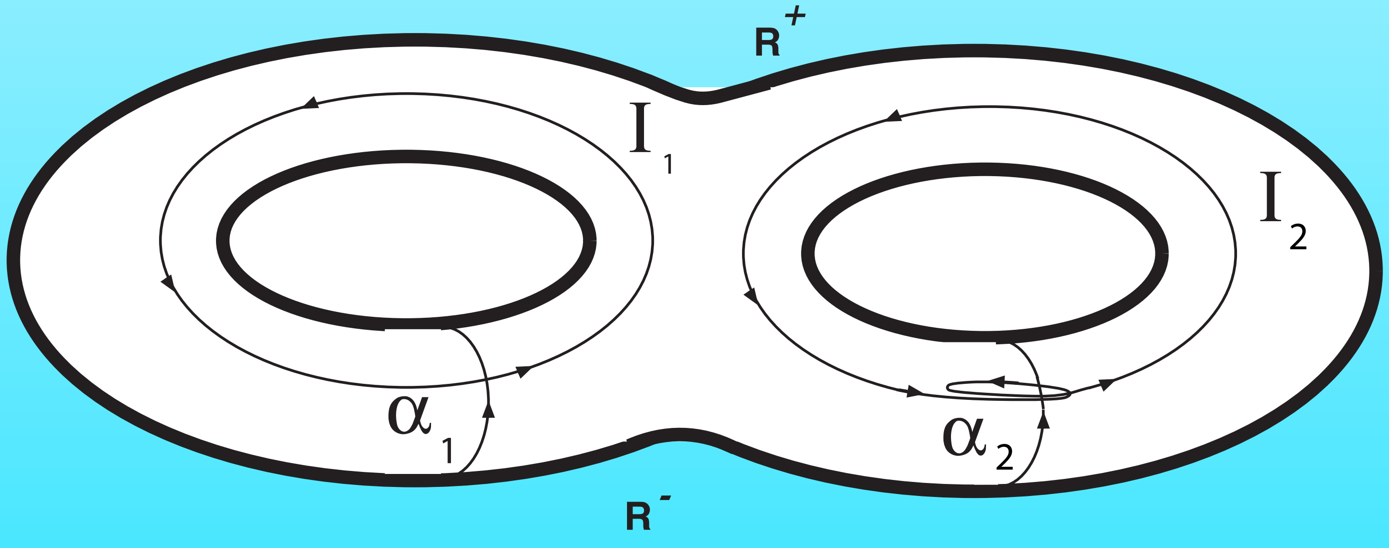
Analytical Consideration
of
Edge states in Graphene

YH, T. Fukui & H. Aoki, Phys. Rev. B74, 205414 (2006)

- ☆ *Followed by the discussion on a square lattice*

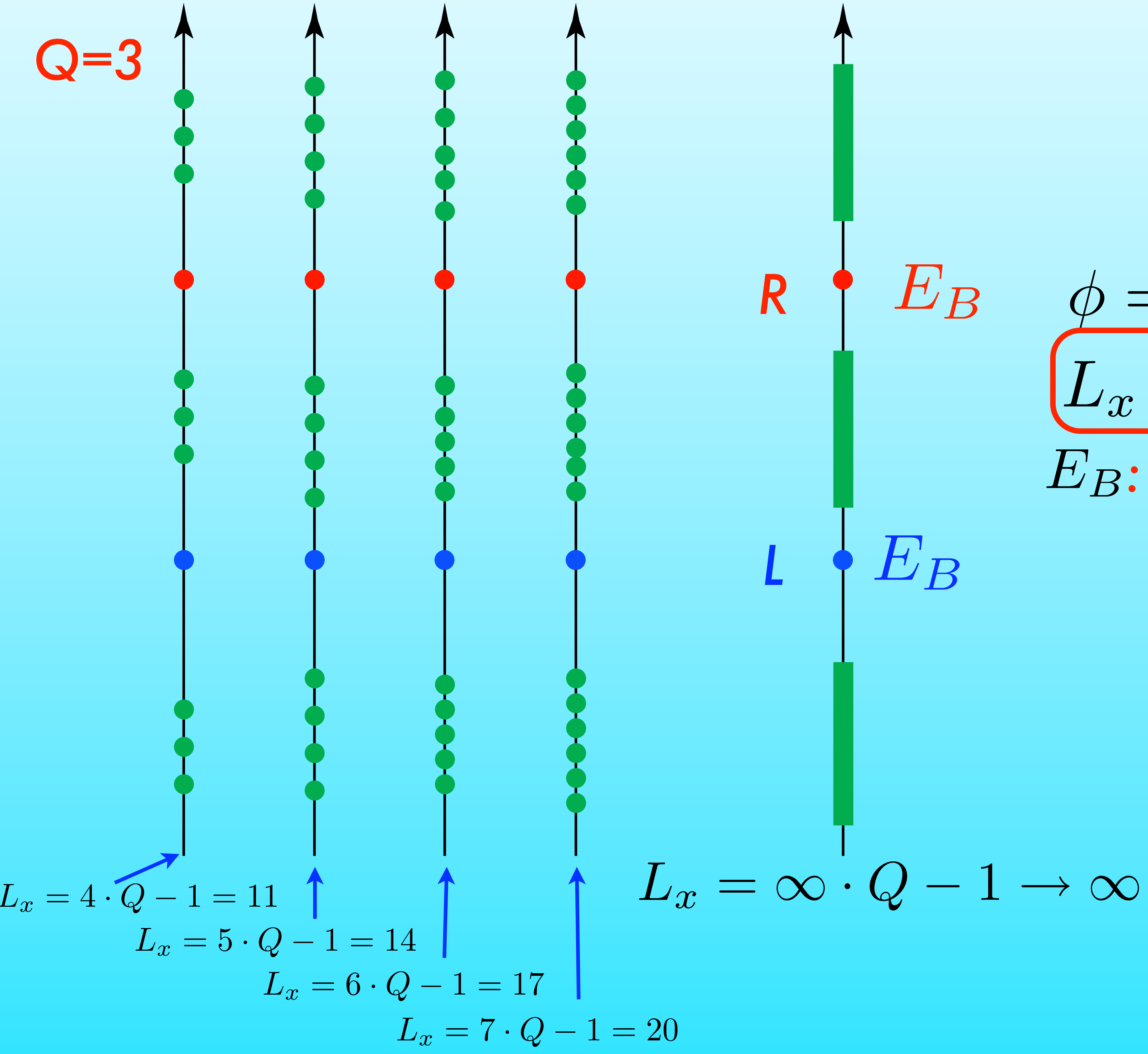
Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)

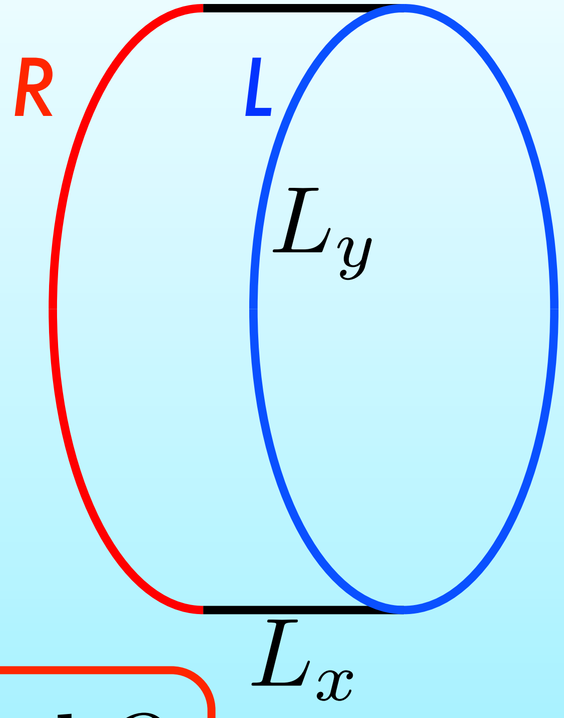




Width (L_x) dependence of the spectrum

☆ 1D system with Boundaries of length L_x





$$\phi = P/Q$$

$$\boxed{L_x \equiv -1, \quad \mathrm{mod}\, Q}$$

E_B: independent of L_x

Q=2q: graphene

Infinite size limit
with Boundaries

Edge State and Bloch State

☆ reduced 1D system and transfer matrix

$$H=\sum_{k_y} H_{\rm 1D}(k_y)$$

Y.H., Phys. Rev. B 48, 11851 (1993)
Phys. Rev. Lett. 71, 3697 (1993)

$$|E,k_y\rangle=\sum_{j_x}\left[\psi_\bullet(E,j_x,k_y)c_\bullet^\dagger(j_x,k_y)|0\rangle+\psi_\circ(E,j_x,k_y)c_\circ^\dagger(j_x,k_y)|0\rangle\right],$$

$$H_{\rm 1D}(k_y)|z,k_y\rangle=z|z,k_y\rangle,\ \ z=E$$

$$M_{\circ\bullet}(j_x)=\begin{pmatrix}\frac{E}{t^*_{\circ\bullet}(j_x)} & -\frac{t_{\bullet\circ}(j_x-1)}{t^*_{\circ\bullet}(j_x)}\\ 1 & 0\end{pmatrix}$$

Transfer matrix $\boldsymbol{\psi}(j_x+1)=M_{\rm t}(j_x)\boldsymbol{\psi}(j_x)$

$$M_{\bullet\circ}(j_x)=\begin{pmatrix}\frac{E}{t^*_{\bullet\circ}(j_x)} & -\frac{t_{\circ\bullet}(j_x)}{t^*_{\bullet\circ}(j_x)}\\ 1 & 0\end{pmatrix}$$

$$t_{\circ\bullet}(j_x,k_y)=t\left(1+e^{ik_y-i2\pi\phi j_x}\right)$$

$$t_{\bullet\circ}(j_x,k_y)=t\left[1+(t'/t)e^{ik_y-i2\pi\phi(j_x+1/2)}\right]$$

$$\boldsymbol{\psi}(j_x)=\begin{pmatrix}\psi_\bullet(j_x)\\ \psi_\circ(j_x-1)\end{pmatrix}\qquad M_{\rm t}(j_x)=M_{\bullet\circ}(j_x)M_{\circ\bullet}(j_x)$$

How these two are related ??

Bloch State

$$\boldsymbol{\psi}_B(q)=M\boldsymbol{\psi}_B(0)=\rho\boldsymbol{\psi}_B(0)$$

$$|\rho|=1$$

Edge State

$$\boldsymbol{\psi}_E(0)=\begin{pmatrix}1\\ 0\end{pmatrix},\quad \boldsymbol{\psi}_E(q)=\begin{pmatrix}*\\ 0\end{pmatrix}$$

$$M=M_{\rm t}(q-1)M_{\rm t}(q-2)\cdots M_{\rm t}(0)$$

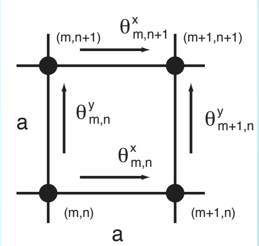
Hofstadter Problem & Graphene under magnetic field

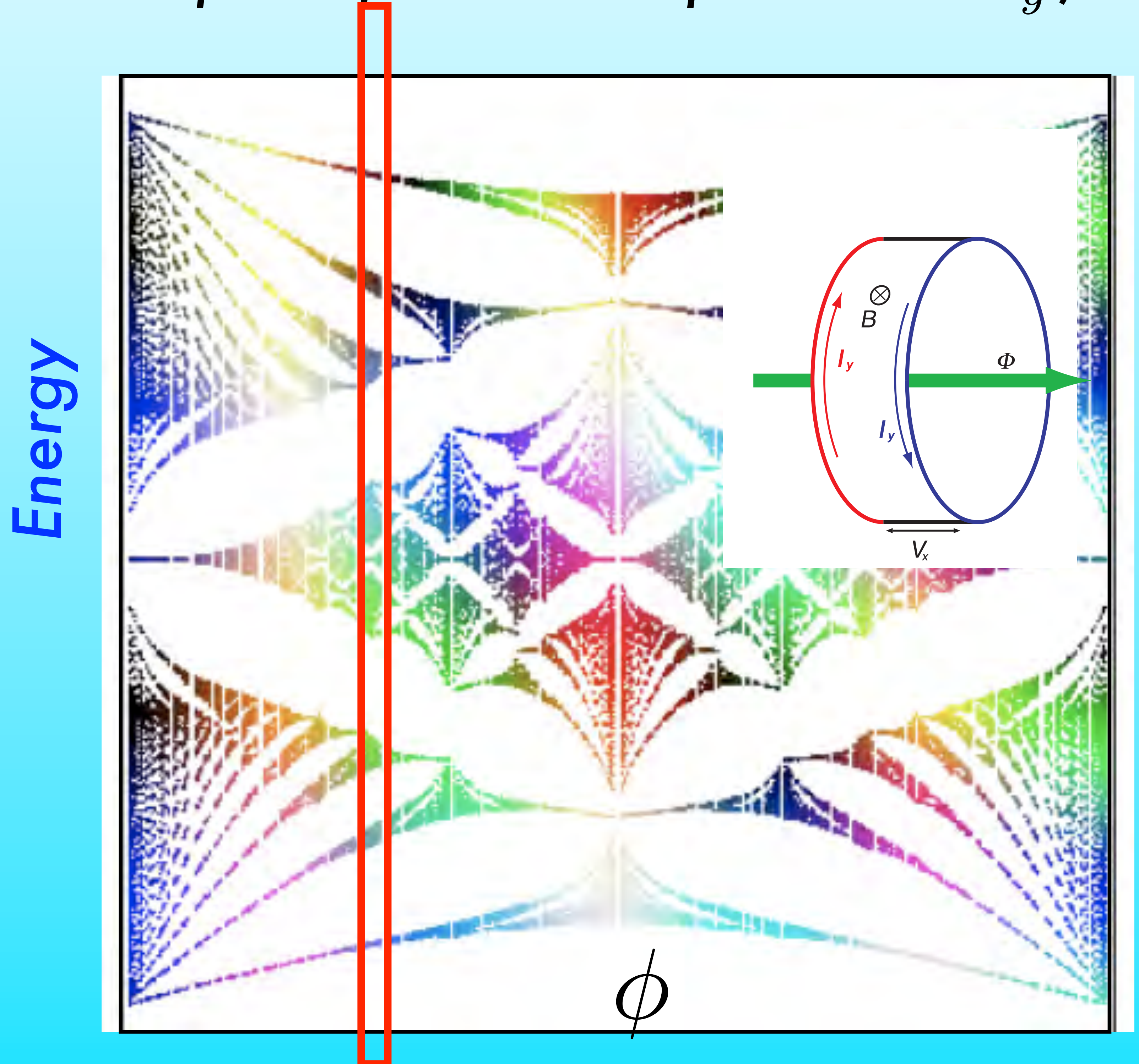
Topol. char. by edges

- In continuum, 2D = $\sum_{k_y}$(1D harmonic oscillators with parameter k_y)

Landau gauge

- Bloch electrons, 2D =$\sum_{k_y}$(1D Harper equation with parameter k_y)





Edge State and Bloch State

- Bloch electrons, 2D = $\sum$(1D Harper problem with parameter k_y)

As for the 1D Harper equation,

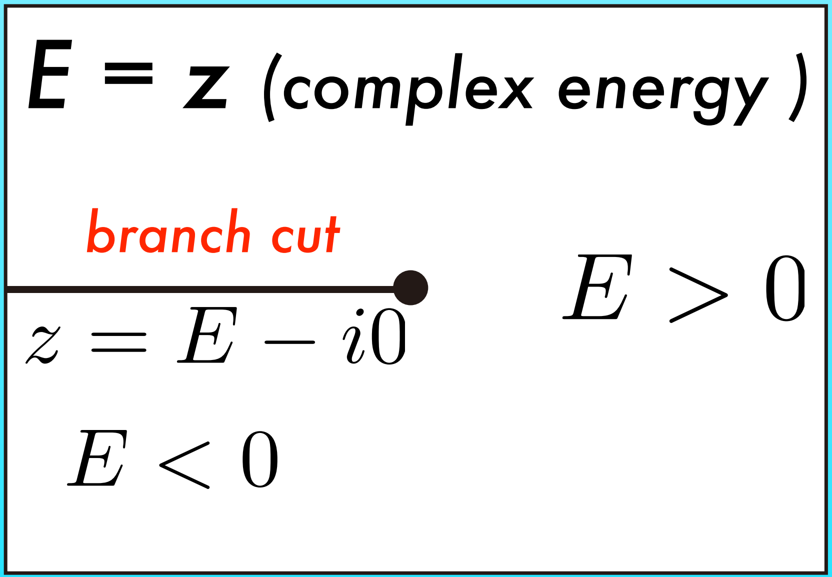
- Edge state : bound state
- Bloch state: scattering state

These two can be treated in a unified way by considering complex energy

standard quantum mechanics

- bound state
- scattering state

$$E = \frac{\hbar^2 k^2}{2m} \begin{cases} < \ 0 & k = i\kappa, \quad \psi \sim e^{-\kappa x} \\ > \ 0 & k \in \mathbb{R}, \quad \psi \sim e^{ikx} \end{cases}$$



E = z (complex energy)

branch cut

$z = E - i0$

$E > 0$

$E < 0$

unified description

$$\psi \sim e^{i\sqrt{2mE}\, x/\hbar}$$

energy of the bound state is in the gap region E<0

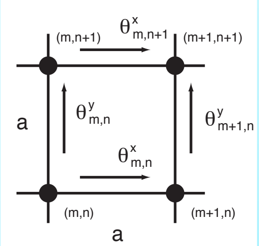
Hofstadter Problem & Graphene under magnetic field

Topol. char. by edges

- In continuum, 2D = $\sum_{k_y}$(1D harmonic oscillators with parameter k_y)

Landau gauge

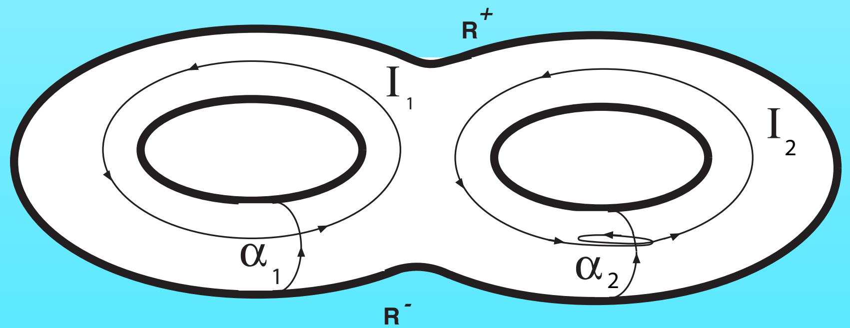
- Bloch electrons, 2D =$\sum_{k_y}$(1D Harper equation with parameter k_y)



Complex energy surface

of

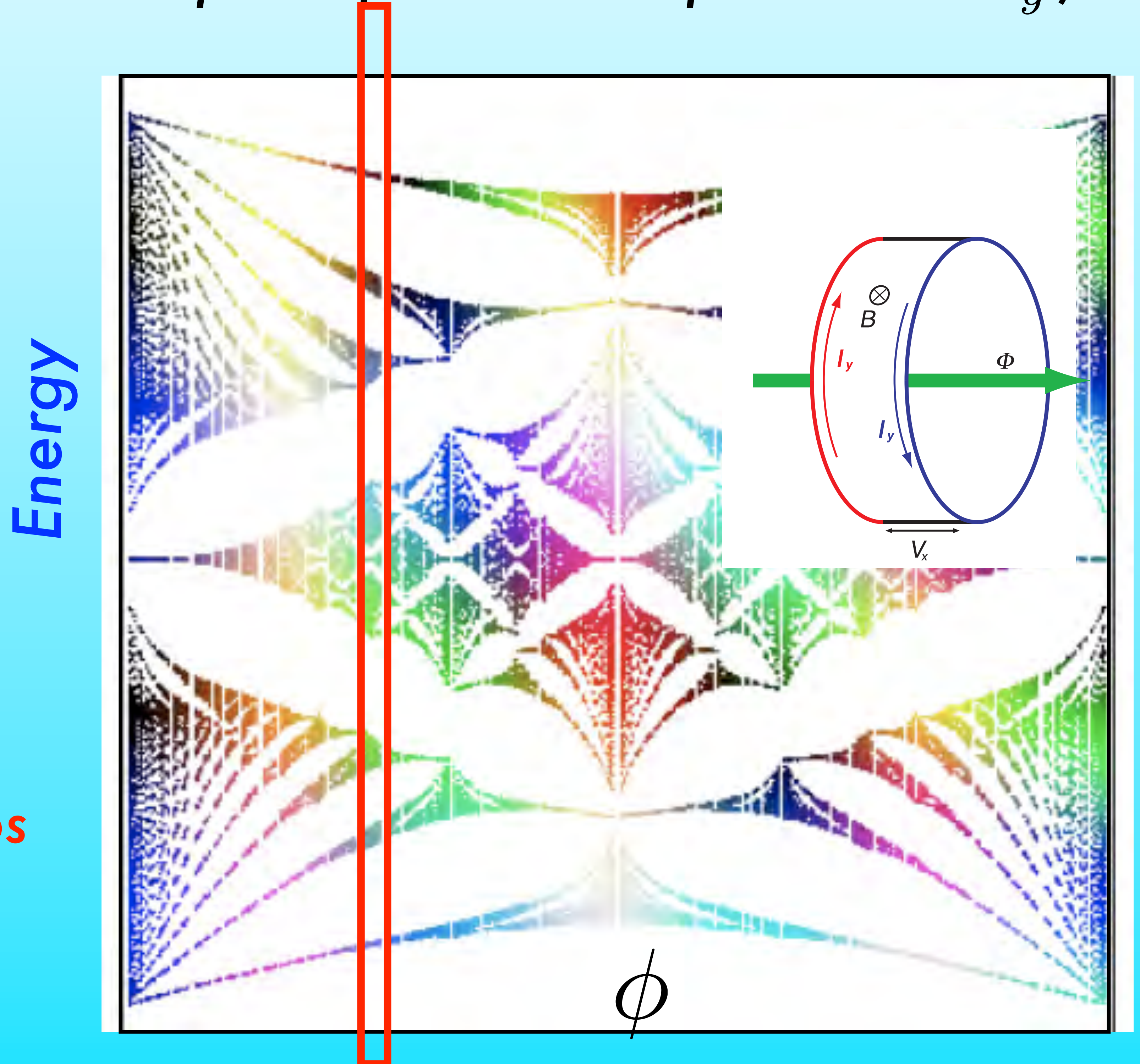
the Harper eq.



q Bands and g=q-1 gaps

Riemann surface

with g handles



Edge states are topological

Quantized Hall conductance by the topological number of edge states

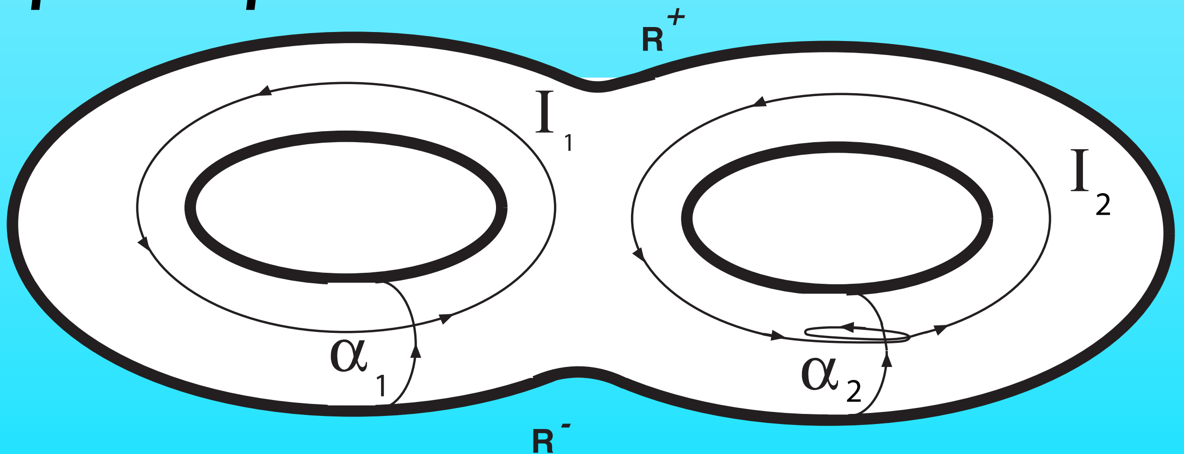
$$\sigma_{xy}{}^{\text{edge}} = \frac{e^2}{h} I_j$$

Topological number

I_j : Winding # of the edge state energy around the handle (energy gap) on the complex energy surface

Complex Energy surface
of Harper eq.

Y. Hatsugai, Phys. Rev. B 48, 11851–11862 (1993)



genus g=q – 1:
number of the gaps

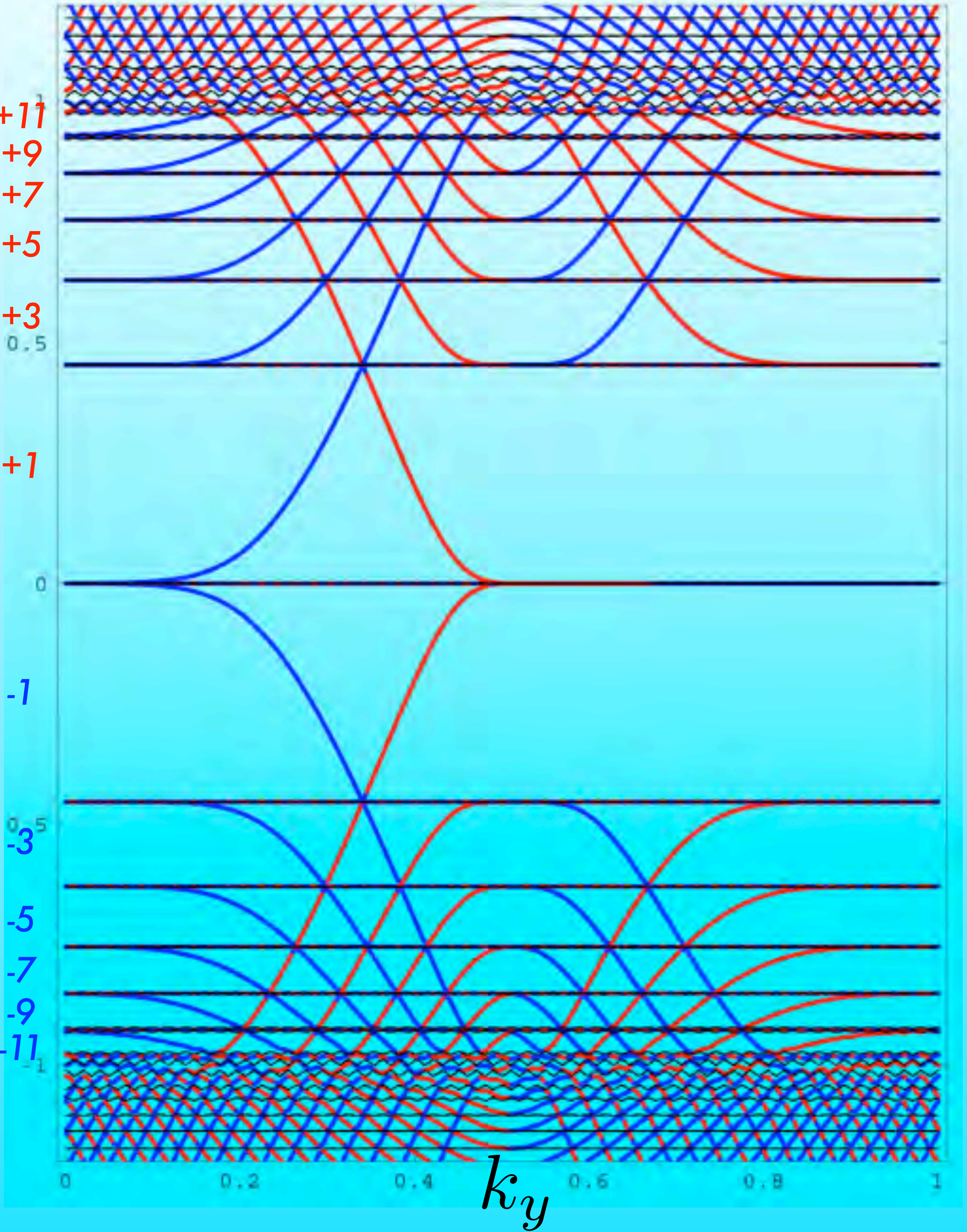
$$\phi = p/q$$

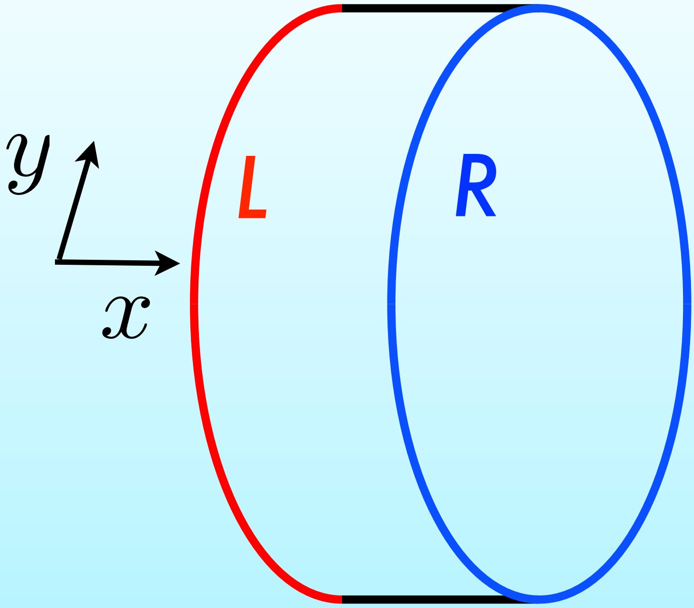
Edge States of Graphene

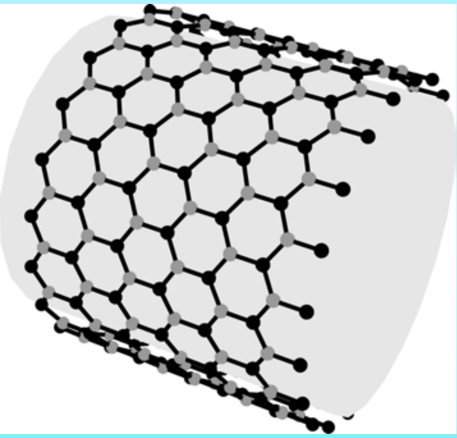
Topol. char. by edges

$\phi = 1/51$

Edge States being consistent with Dirac Type Quantization







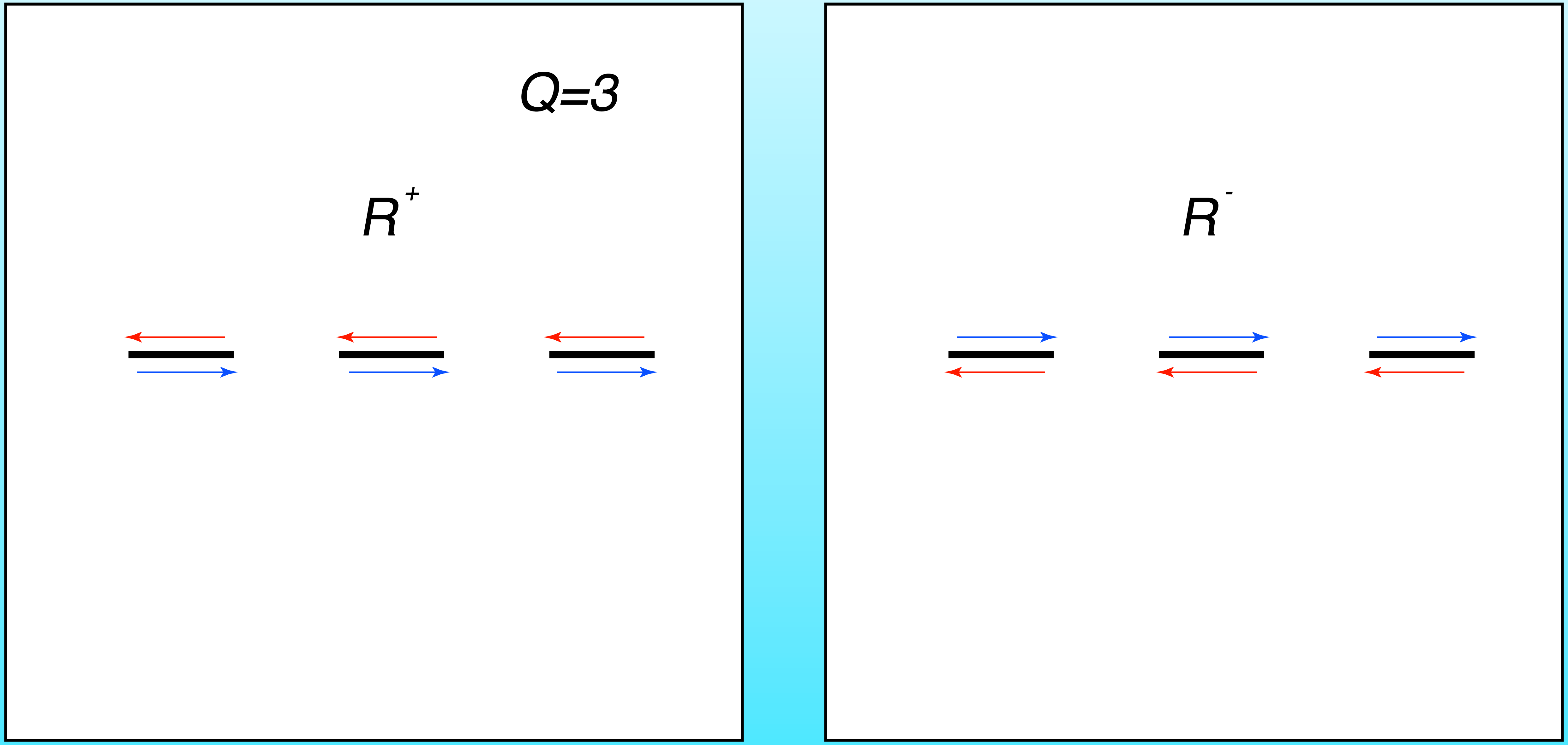
Construction of the Riemann surface

$\phi = 1/3$

$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

- Glue 2 complex planes

Q=3 energy bands: Q=3 branch cuts



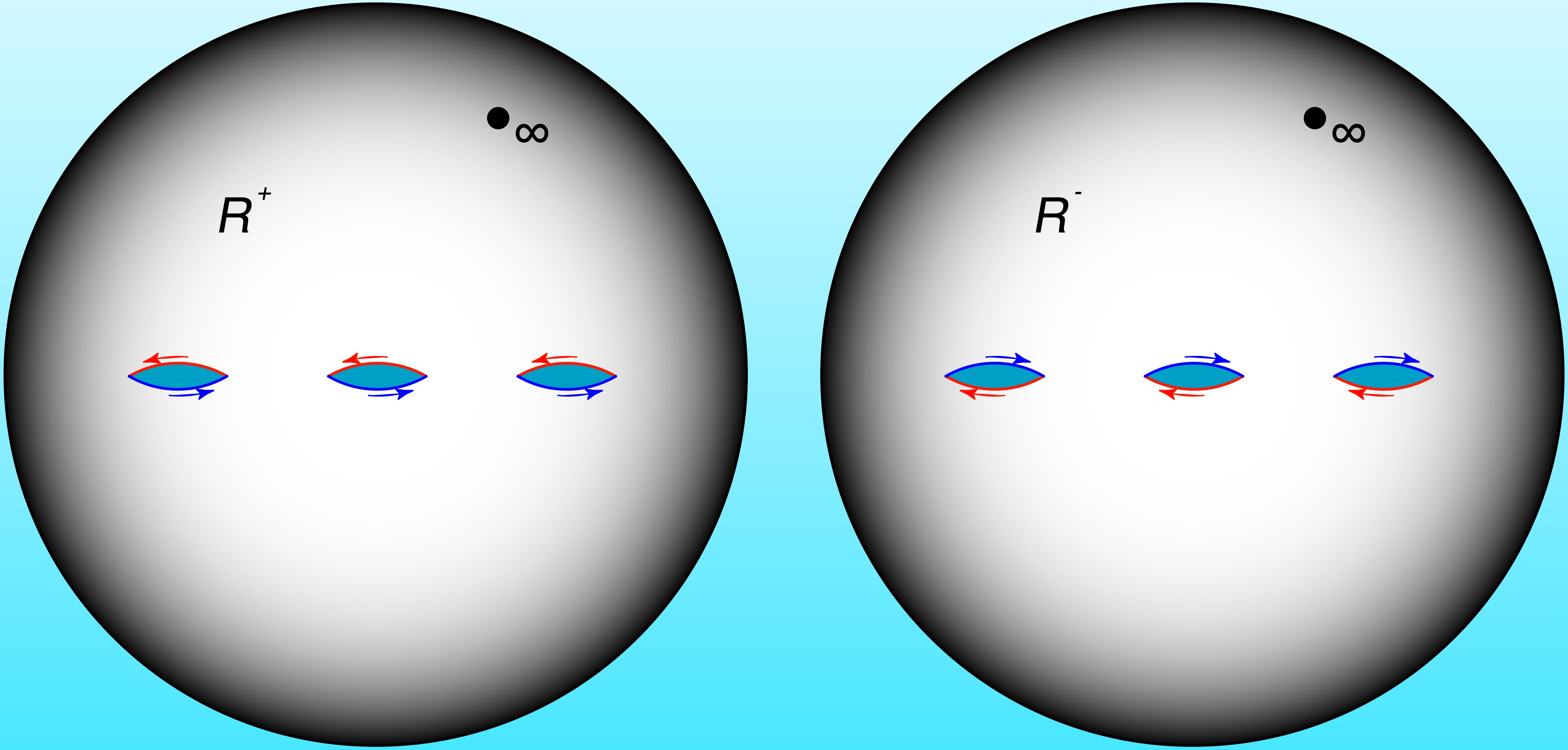
Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$

$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

- ☆ Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts



g=Q-1 holes

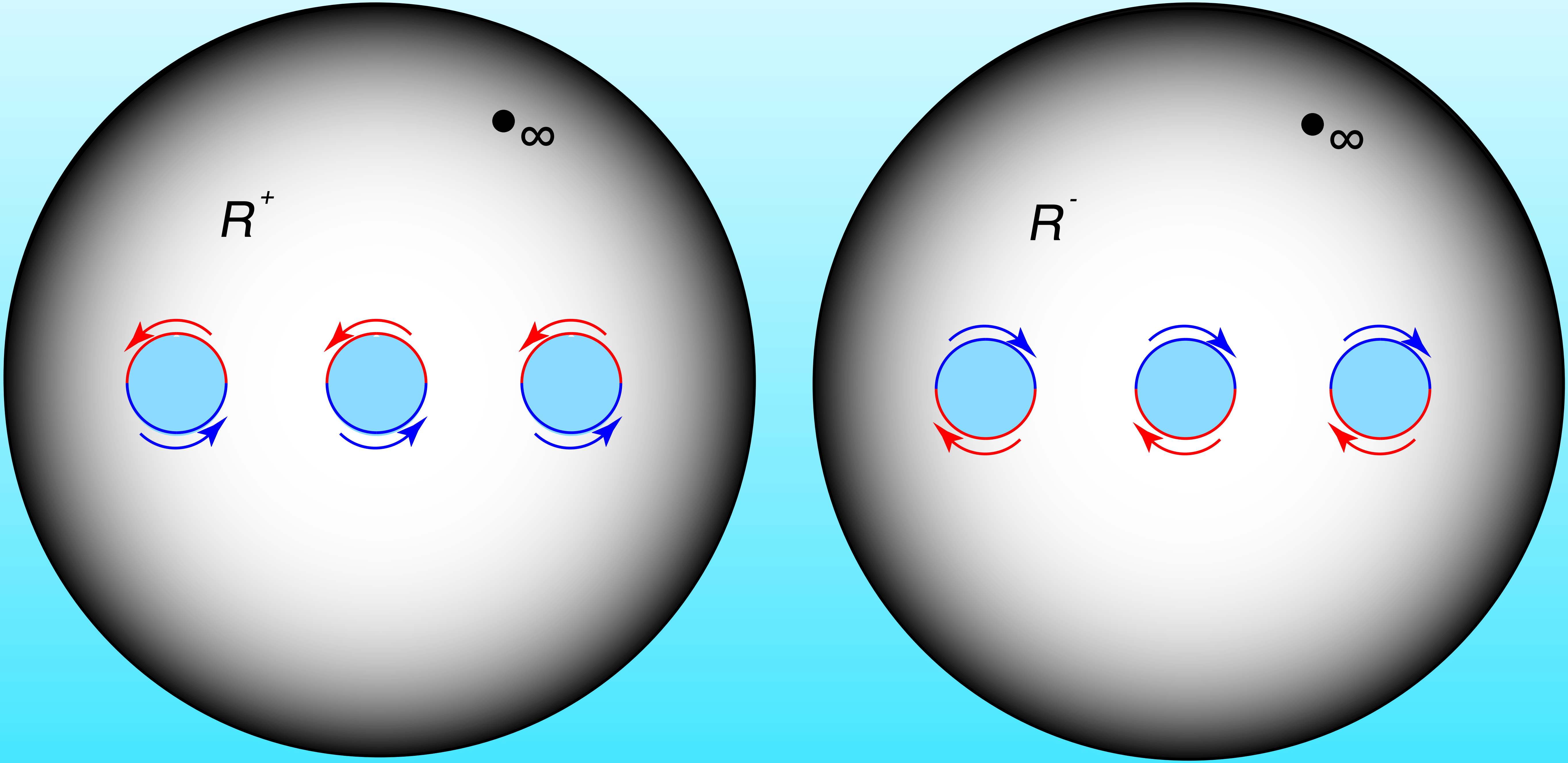
Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$

$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$

- ☆ Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts



g=Q-1 holes

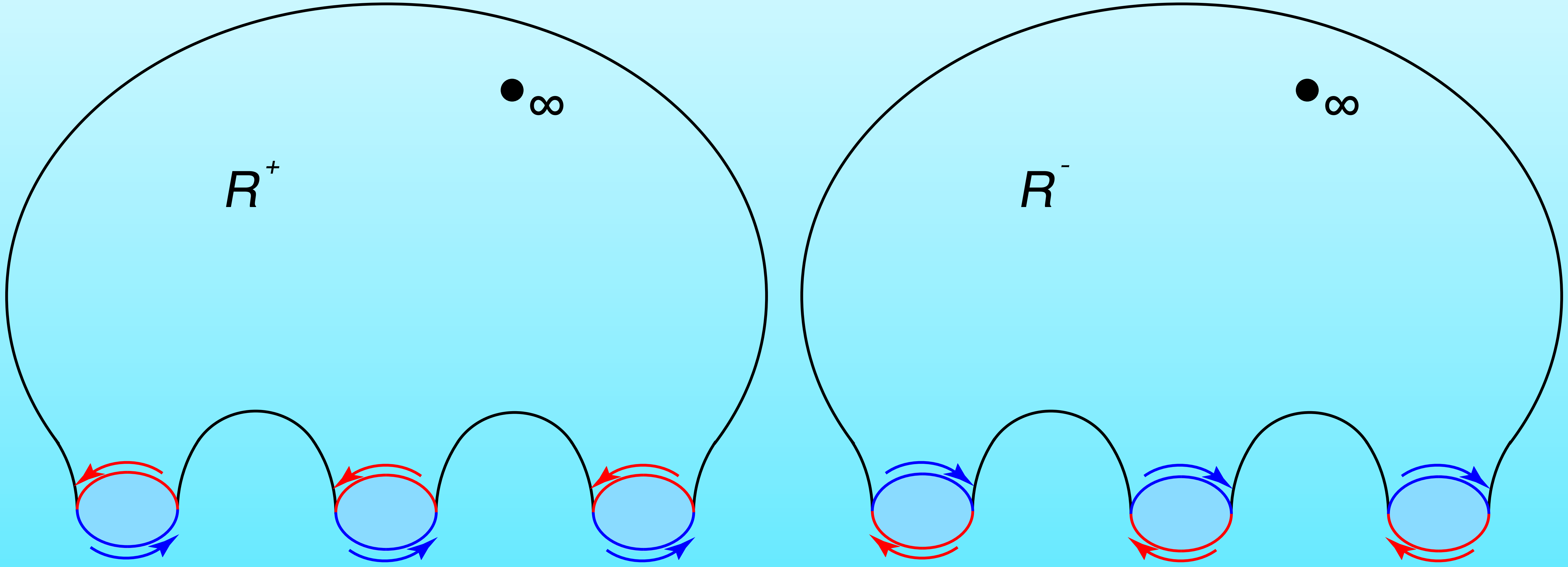
Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$

$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$

- ☆ Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts



g=Q-1 holes

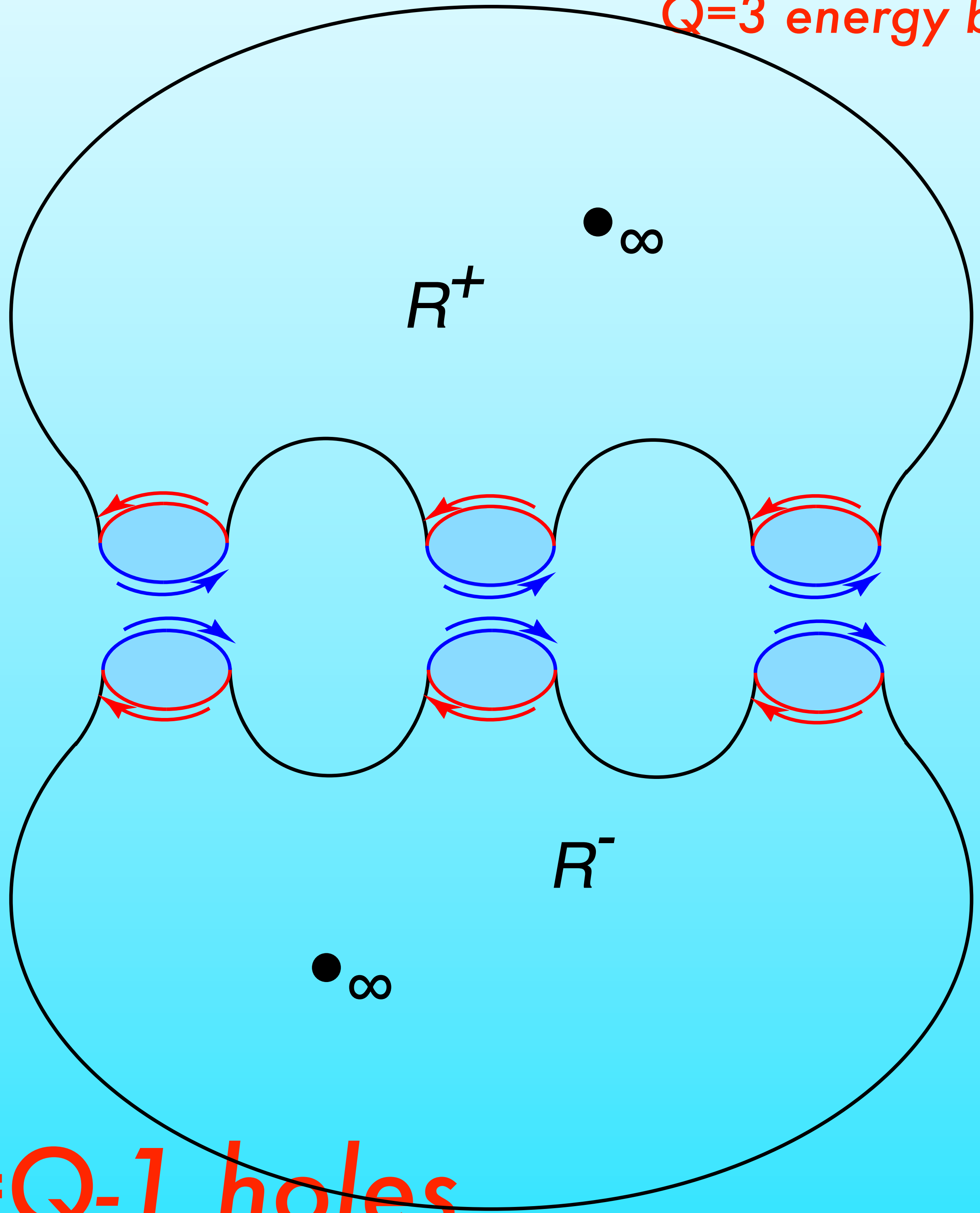
Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$

$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

- Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts



g=Q-1 holes

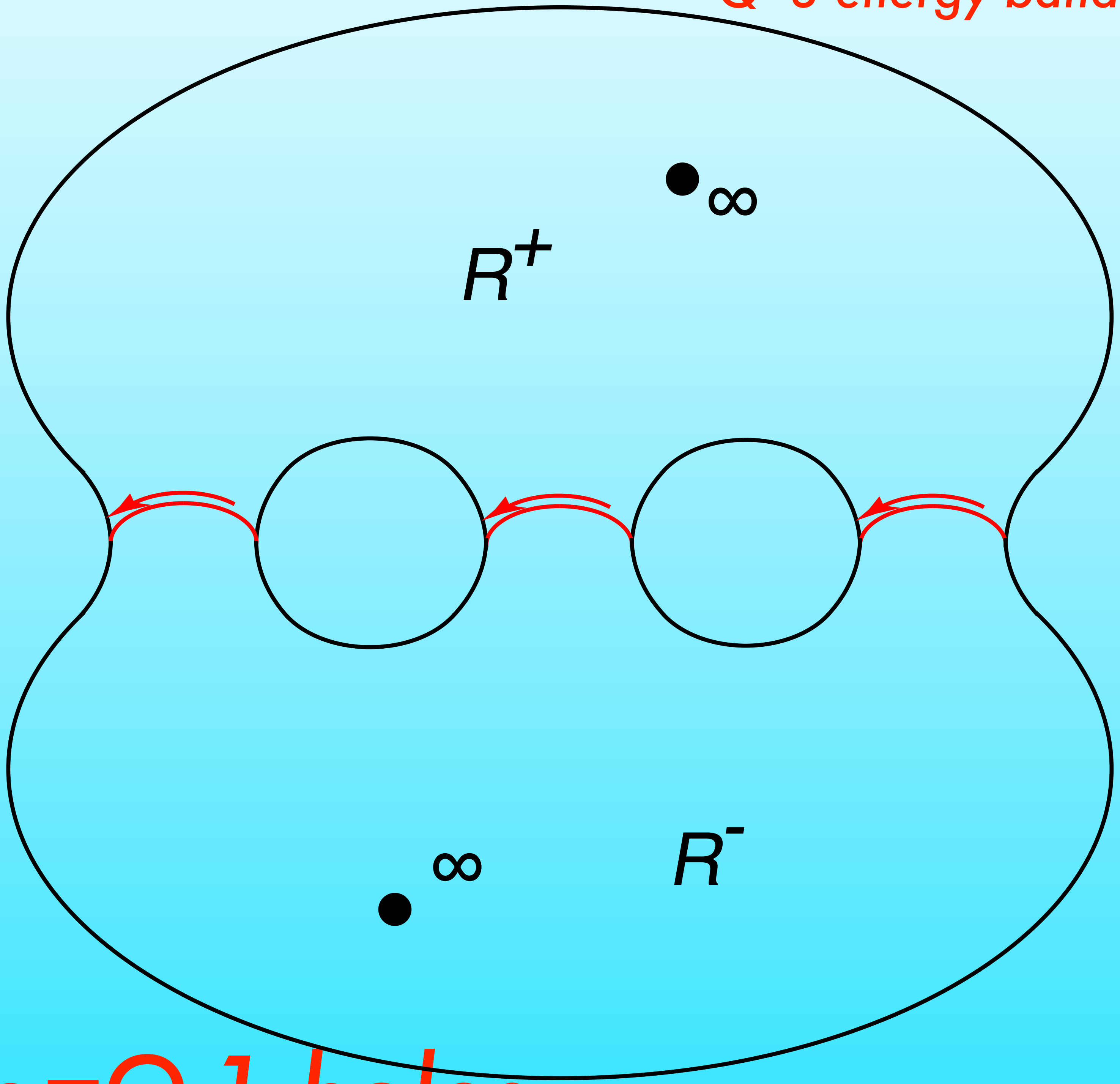
Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$

$$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$$

- ☆ Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts



g=Q-1 holes

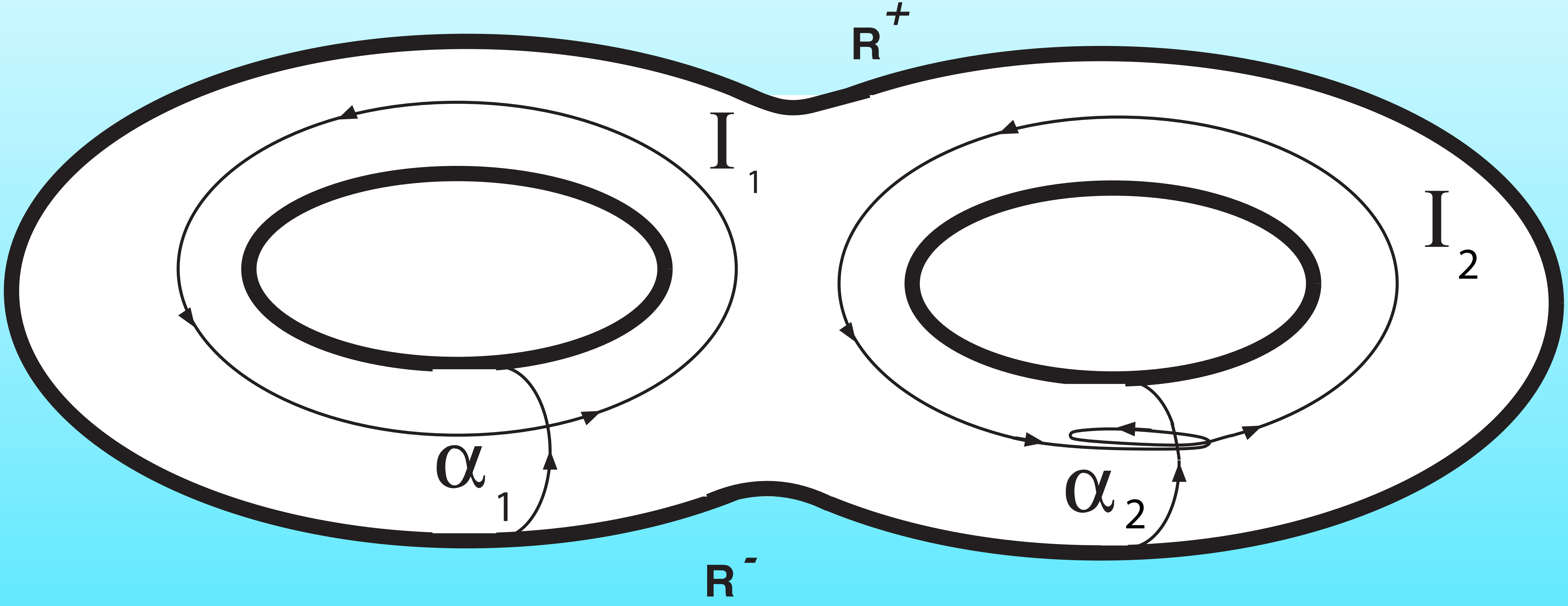
Construction of the Riemann surface

$\Phi = P/Q, \quad Q = 3$

$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2Q-1})(z-\lambda_{2Q})}$

- Glue 2 complex planes with Q branch cuts

Q=3 energy bands: Q=3 branch cuts



g=Q-1 holes

Wave function & Riemann Surface Σ_g

As for fixed k_y of the 1D-Harper systems

☆ Zeros of the Bloch fn. defines the Edge State Energies

Energy bands ⟷ Branch cuts

Energy gaps ⟷ Holes

$\phi = P/Q$

$g = Q - 1$

W. fn. is localized at ⟷ The zero of the Bloch fn. is on

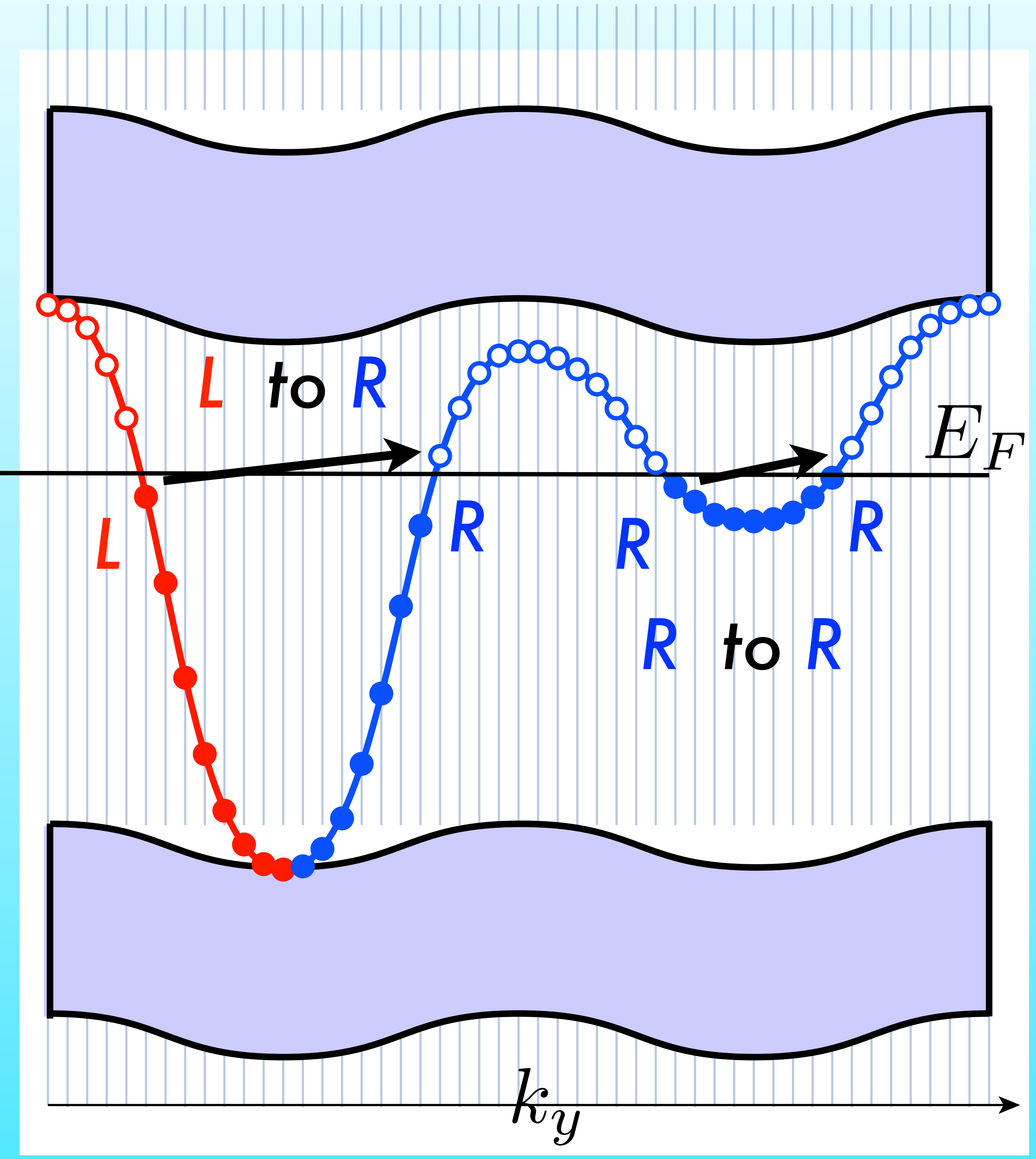
the left edge ⟷ the upper Riemann Surface R^+

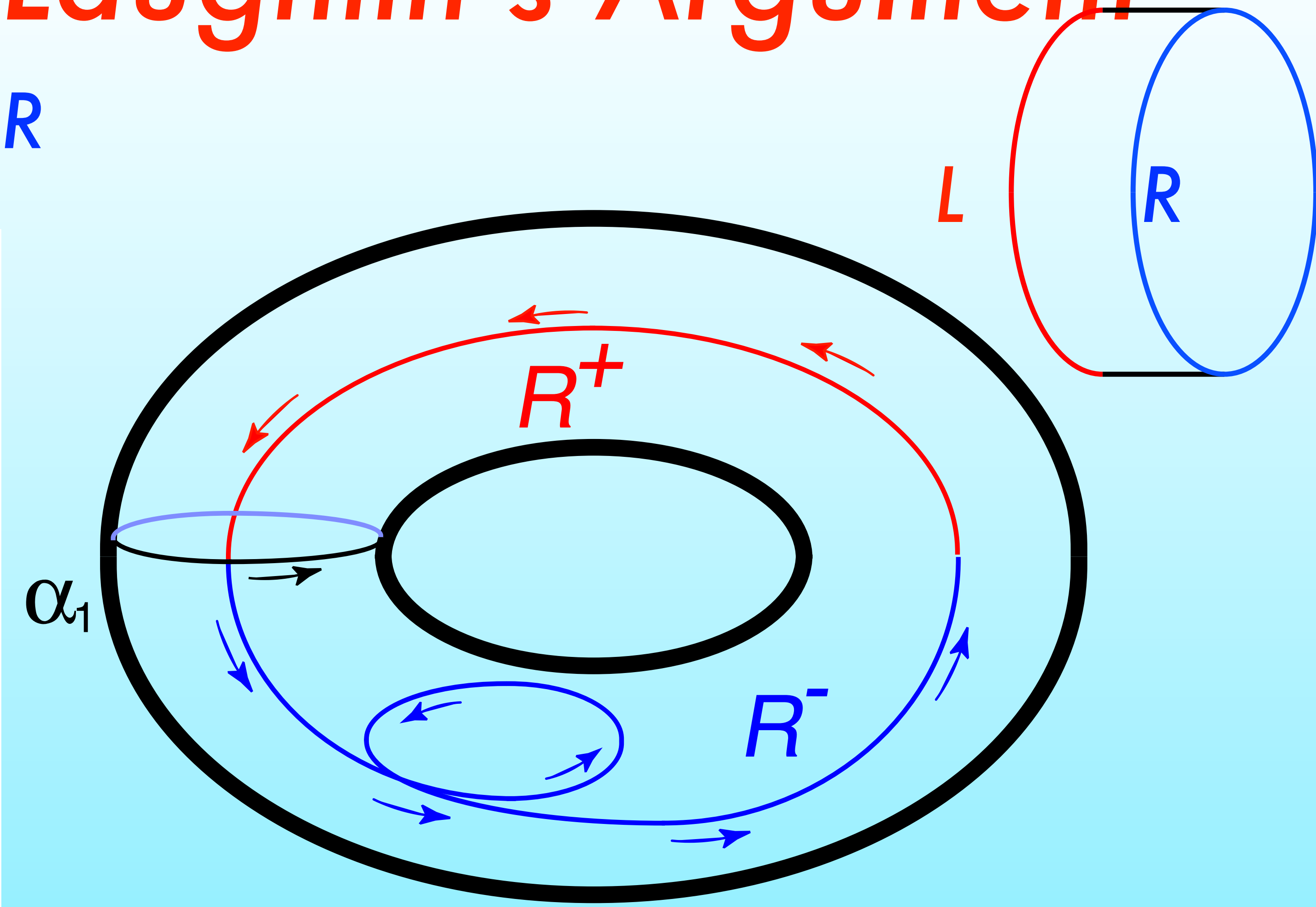
the right edge ⟷ the upper Riemann Surface R^-

☆ Changing $k_y \in [0, 2\pi]$, the zero of the Bloch function in the j-th gap makes a closed loop on Σ_g

Riemann surface & Laughlin's Argument

1 state is carried from the L to R





$$I(\alpha_j, L^j_{\mathrm{edge}}) = +1,\ \ j = 1$$

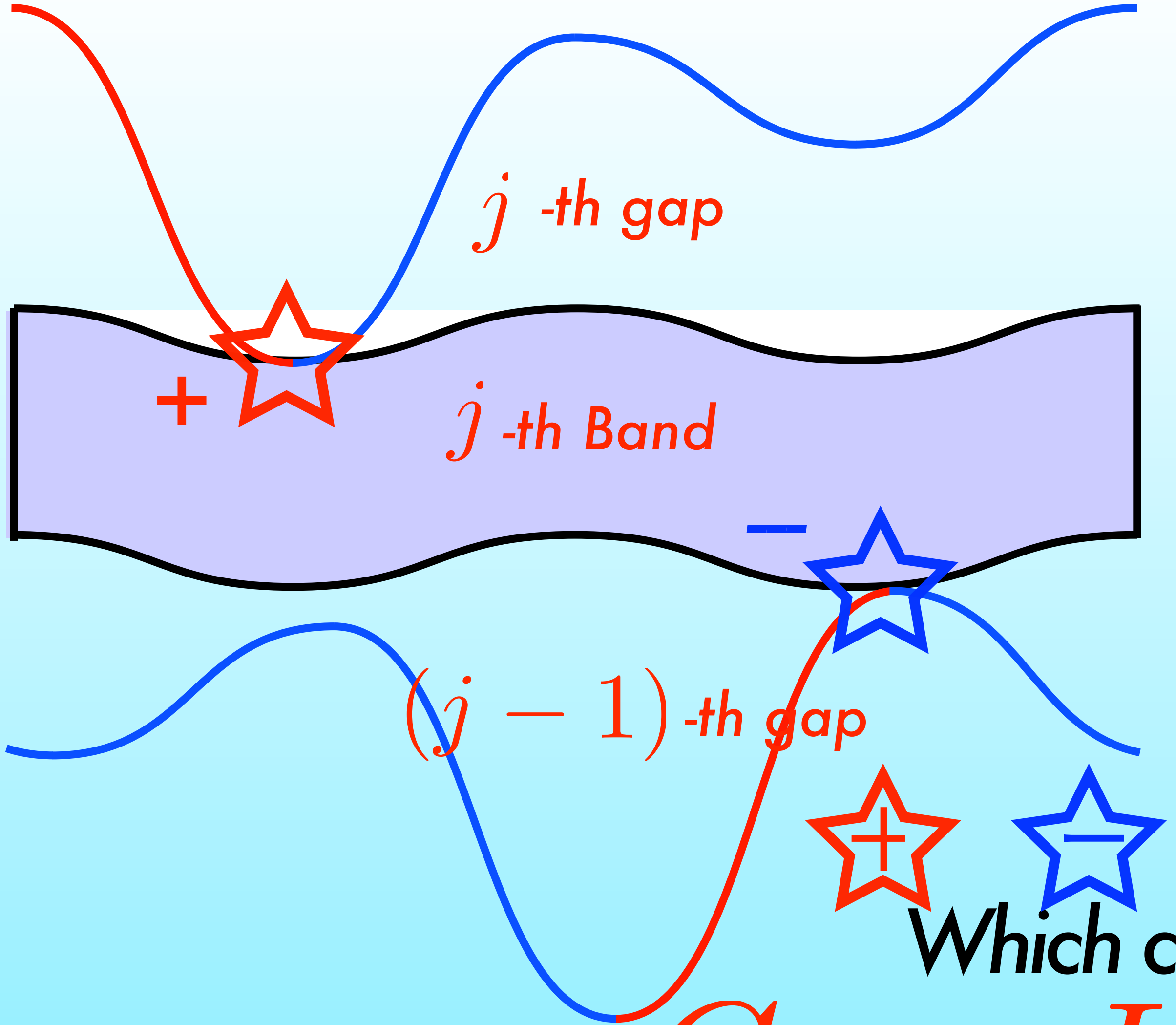
Winding number

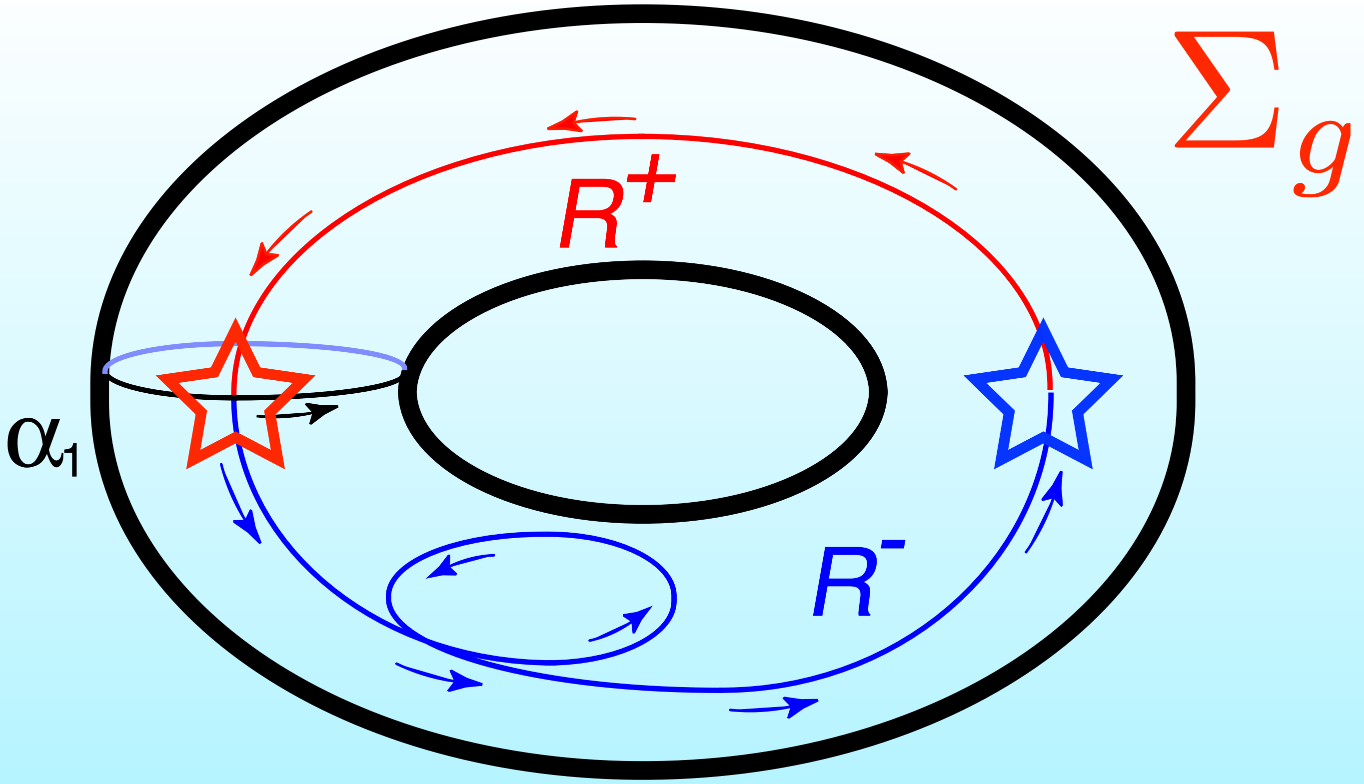
or

Intersection number with canonical loop

Topological number

Y.H., Phys. Rev. B 48, 11851 (1993)





Y.H., Phys. Rev. Lett. 71, 3697 (1993)

Which contribure to the Chern number of the Bulk

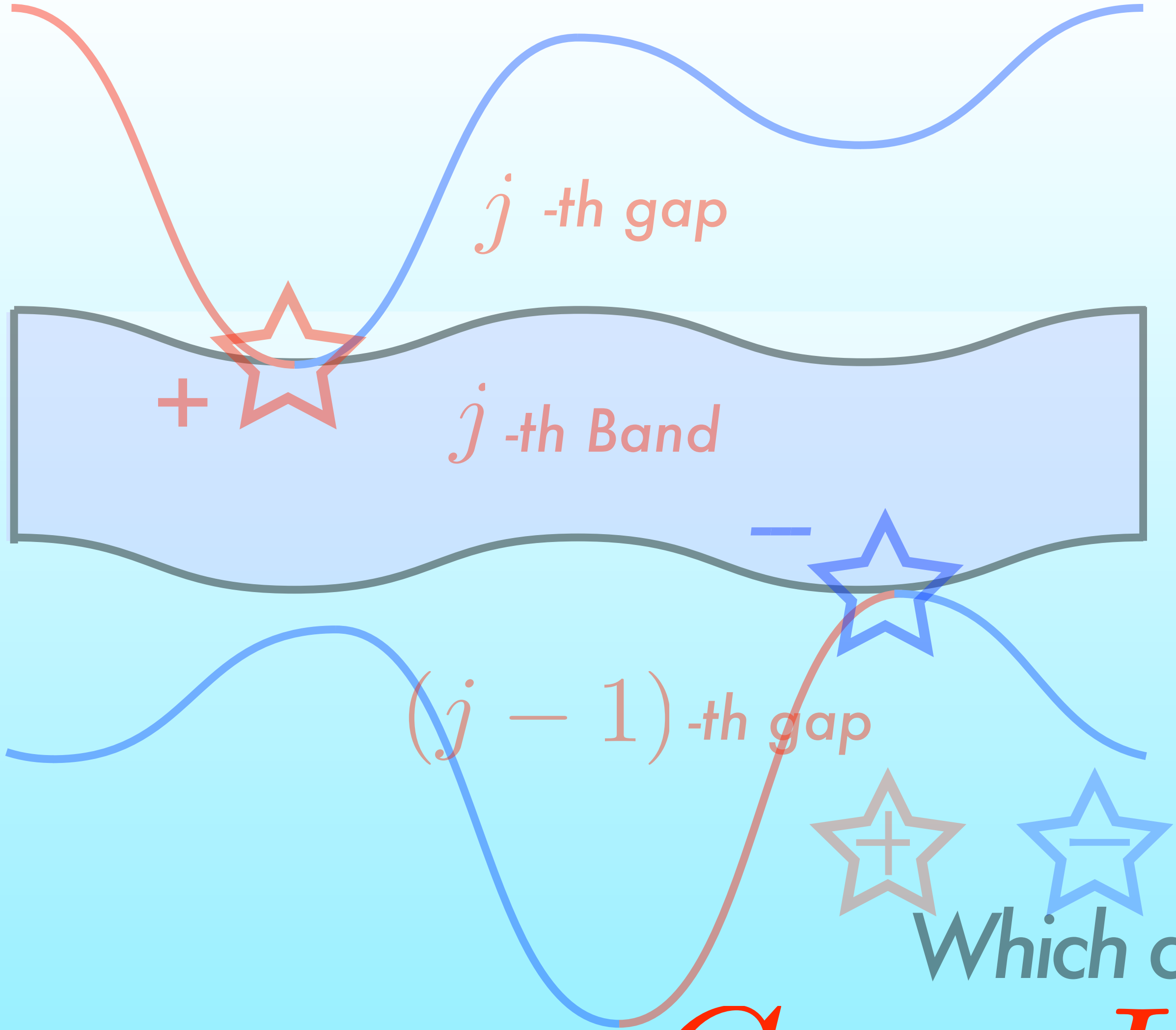
$$C_j = I_j - I_{j-1}$$

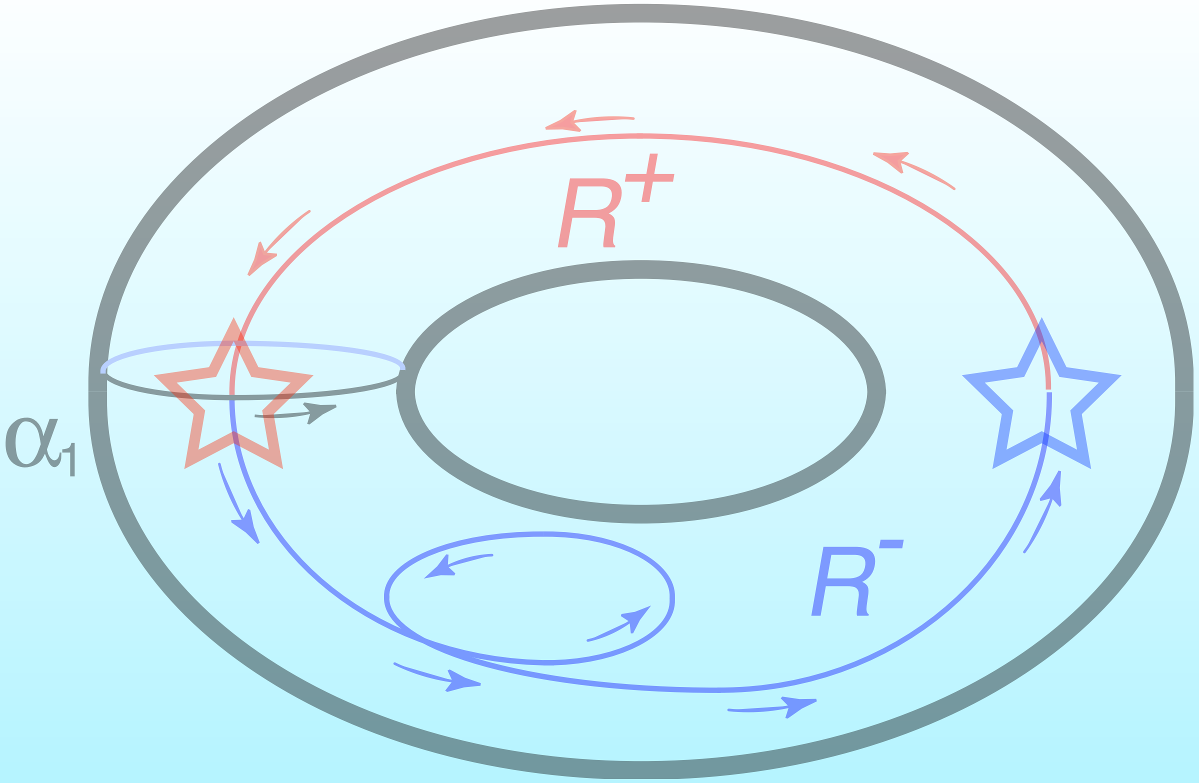
Chern # = winding # Difference between the neighboring gaps

$$\sum_{j=1,\cdots\ell} C_j = I_\ell, \ (\because I_0 = 0)$$

$$\boxed{{\sigma_{xy}}^{\text{bulk}} = {\sigma_{xy}}^{\text{edge}}}$$

Bulk-Edge Correspondence in their topological numbers





Y.H., Phys. Rev. Lett. 71, 3697 (1993)

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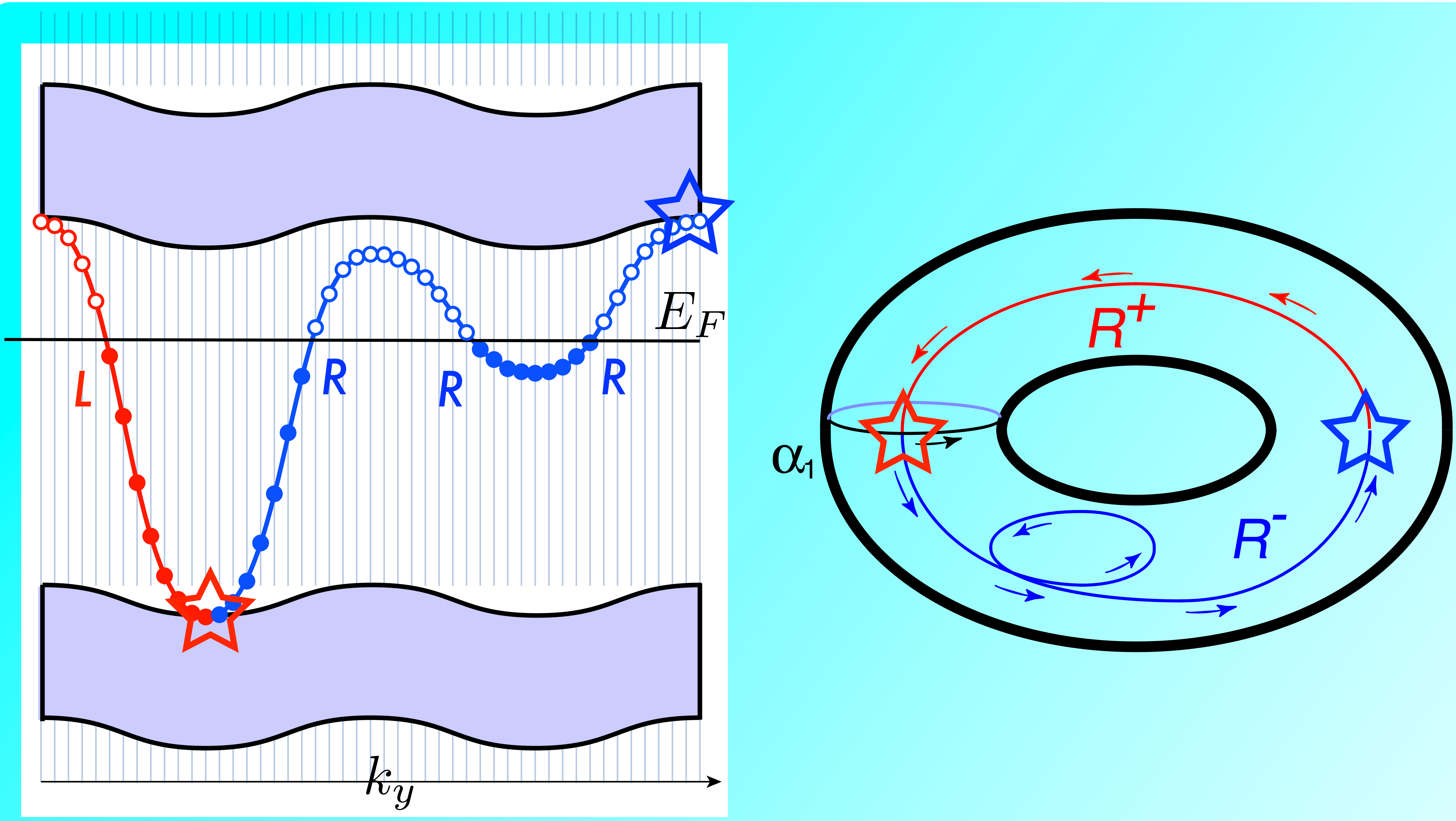
Bulk-Edge Correspondence in their topological numbers



Y.H., Phys. Rev. Lett. 71, 3697 (1993)

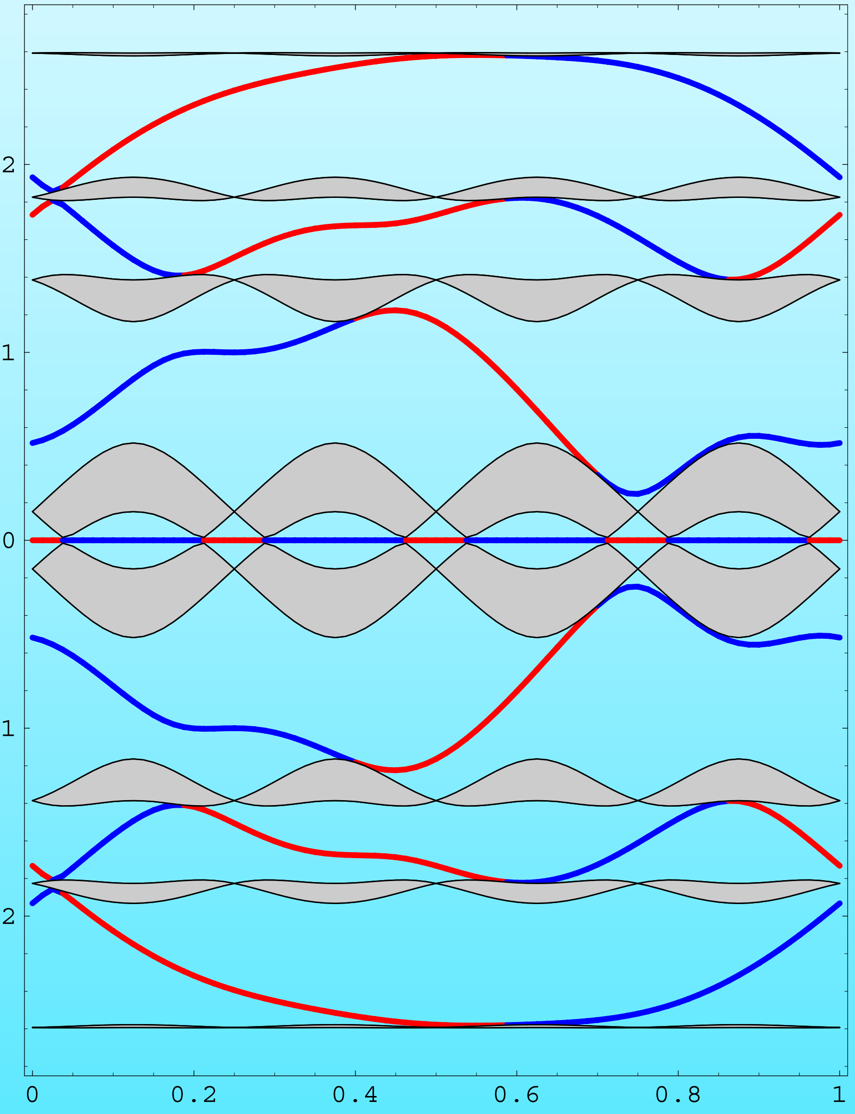
☆ ☆ The touching point makes a vortex in the eneryg band

Which contribute to the Chern number of the Bulk

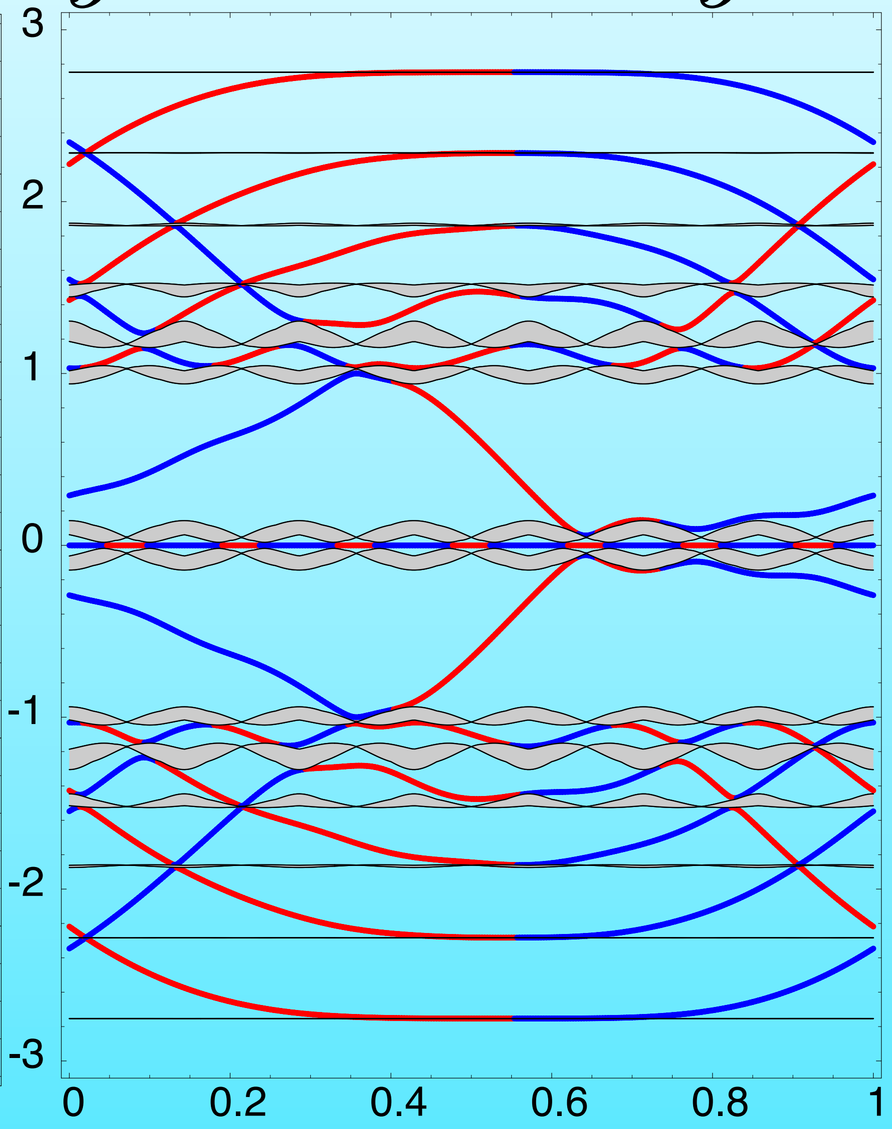
$$C^j_{\mathrm{FS}} = I(\alpha_j, C^j_{\mathrm{edge}})$$

Edge states & Intersection number of Edge State Loops

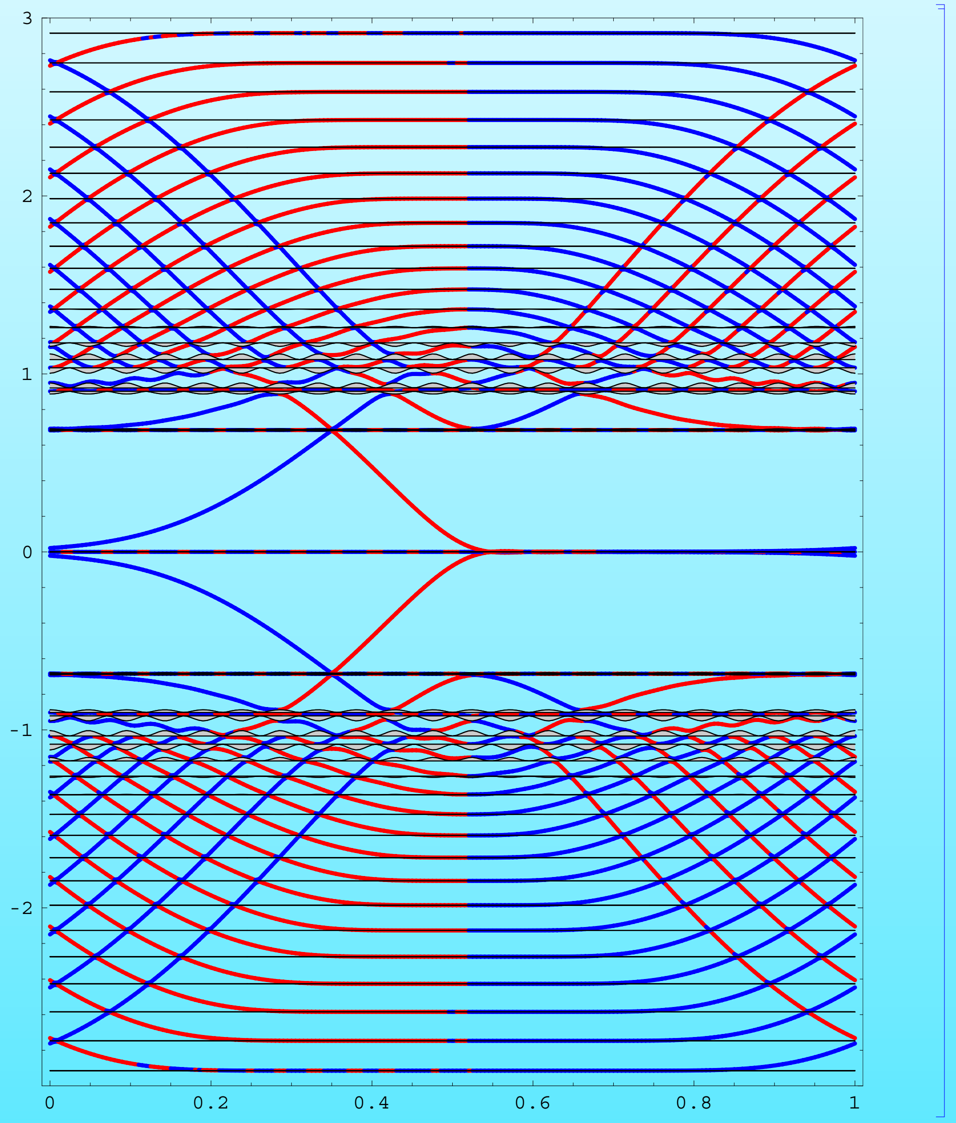
$$\sigma_{xy}^{\text{bulk}} = \sigma_{xy}^{\text{edge}}$$



$\phi = 1/4$



$\phi = 1/7$



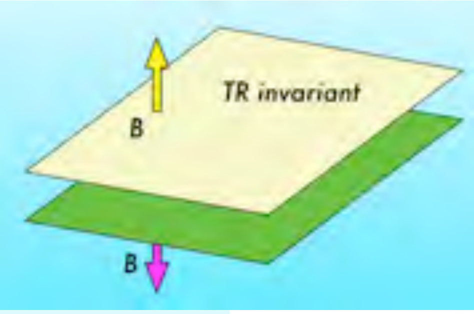
$\phi = 1/21$

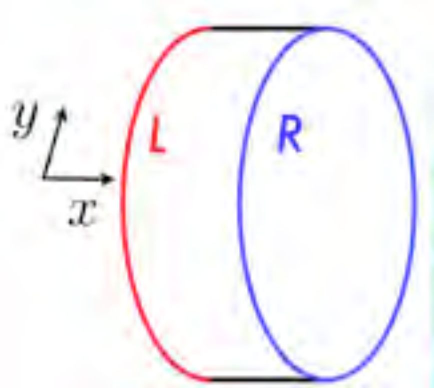
Imagine loops on the Riemann surface

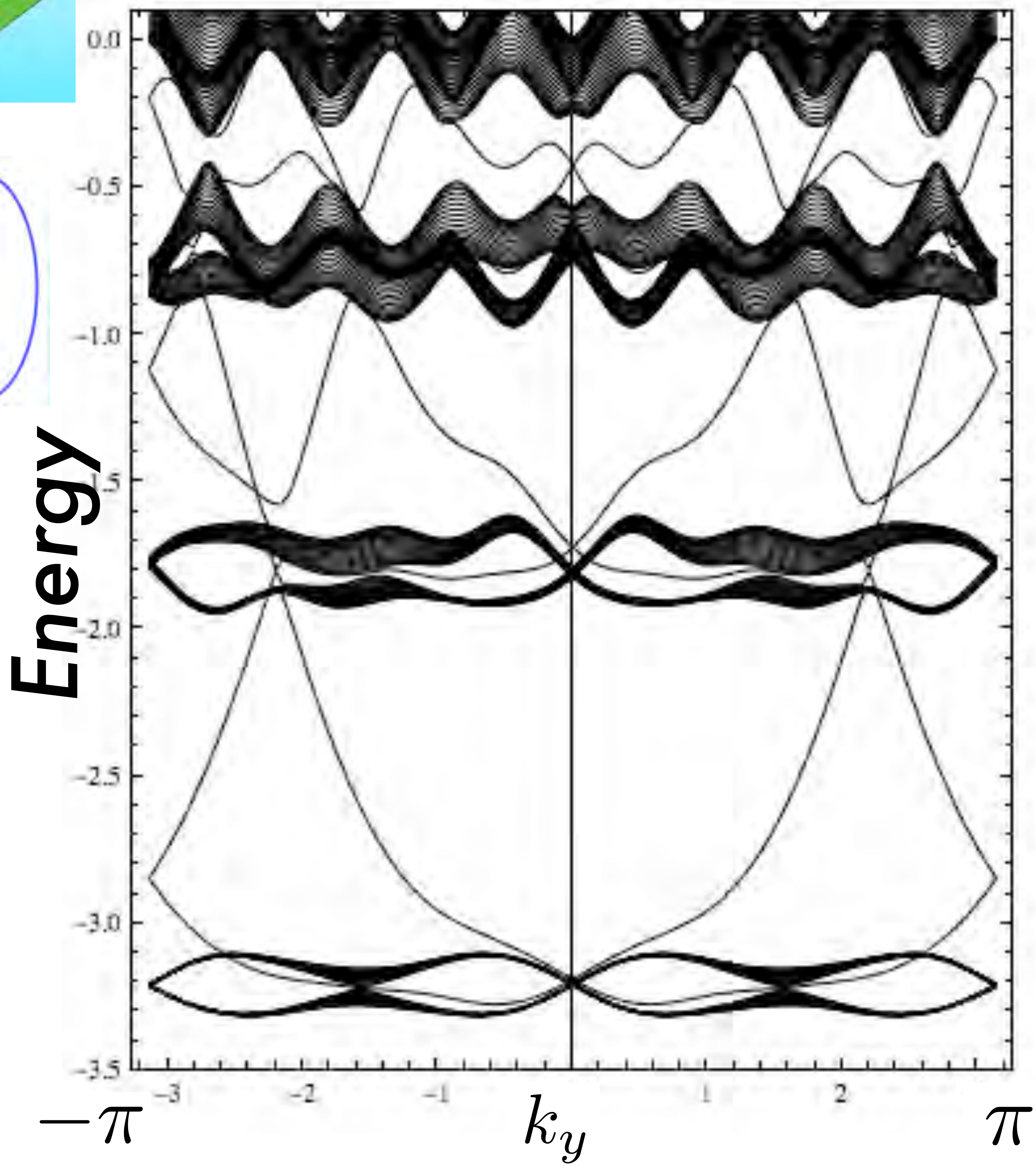
- Edge states of Z_2 topological phase
  - Spin conserved case
  - chiral edge states to helical ones
  - Kramers degeneracy
  - Z2 characterization by the edge states

What's this

2D Quantum Spin Hall state







Edge states are not
chiral, but helical

$$\Theta H(k)\Theta^{-1} \cong H(-k)$$

$$+k \longleftrightarrow -k$$

Not independent
Correlated

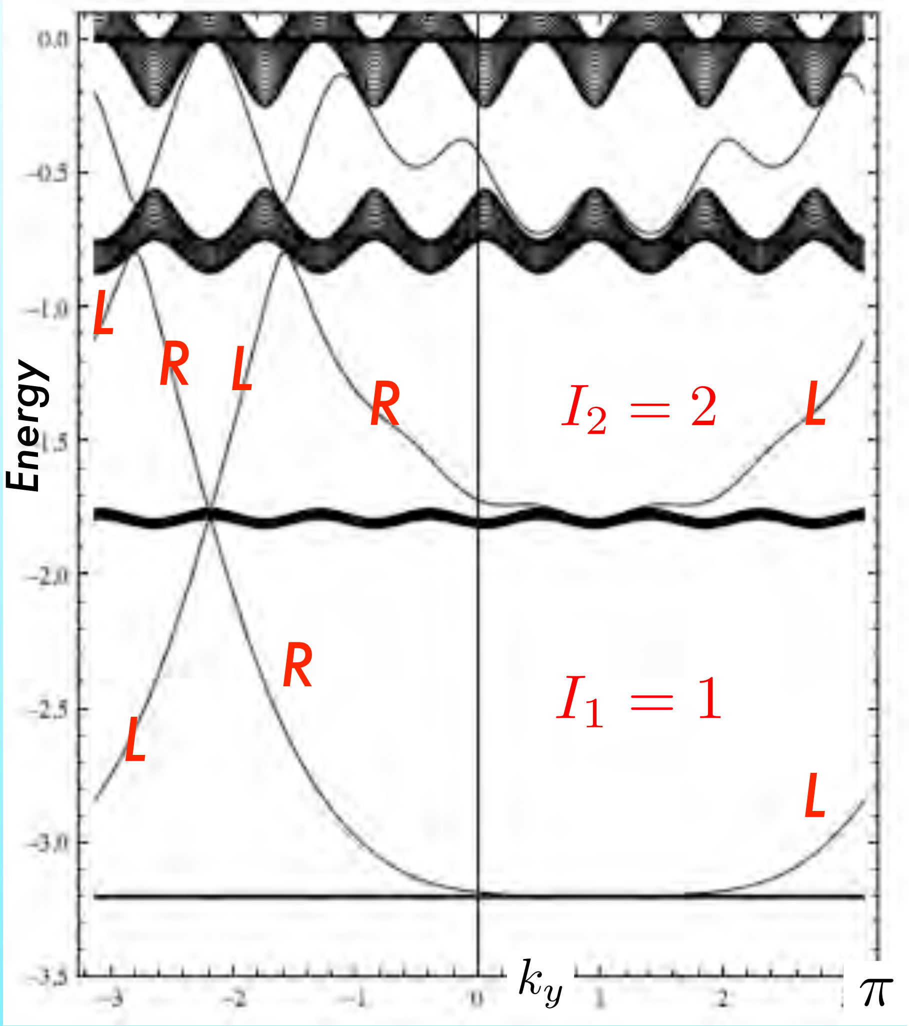
Generically
TR is broken in
momentum space

TR is OK
at special momentum
π and 0

$$H(0) = H(-0)$$
$$H(\pi) = H(-\pi)$$

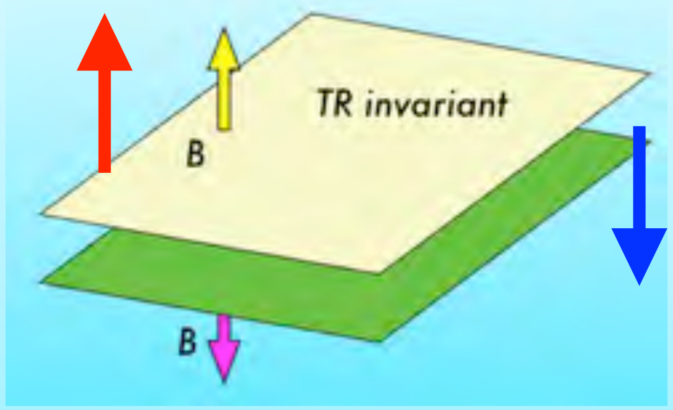
Identification of edge states (QSHE)

upspin

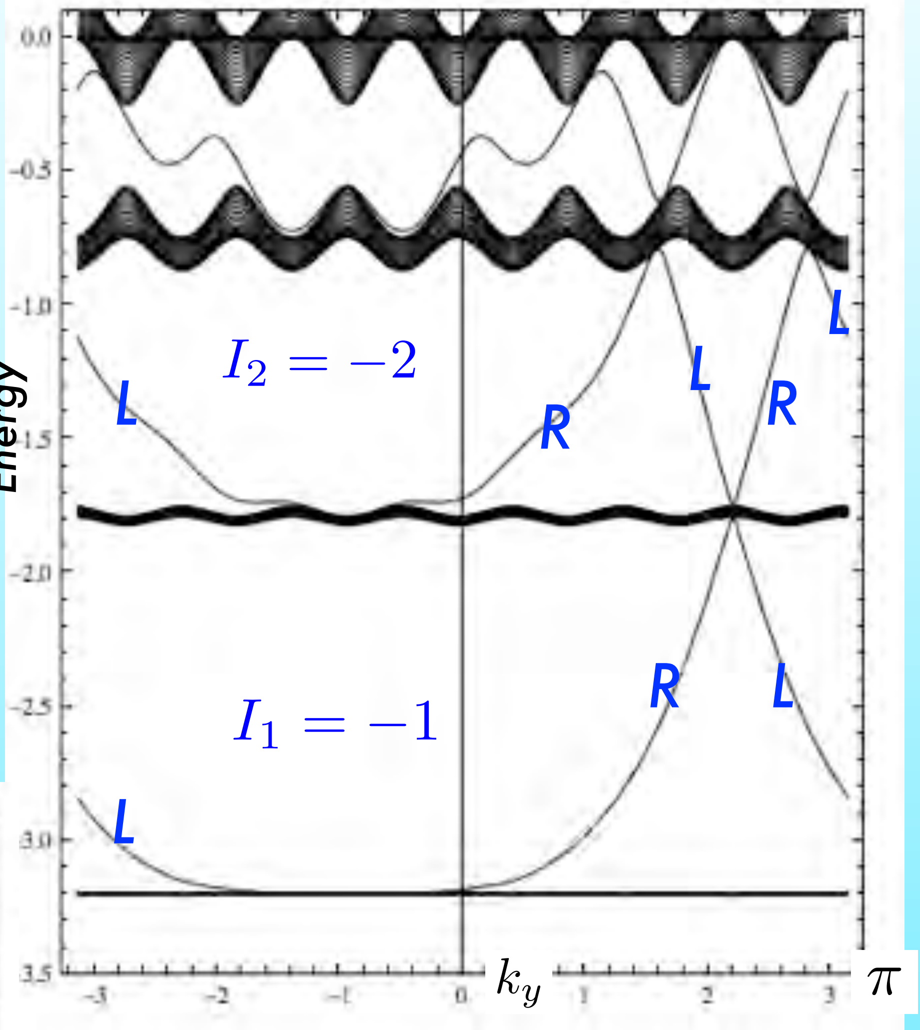


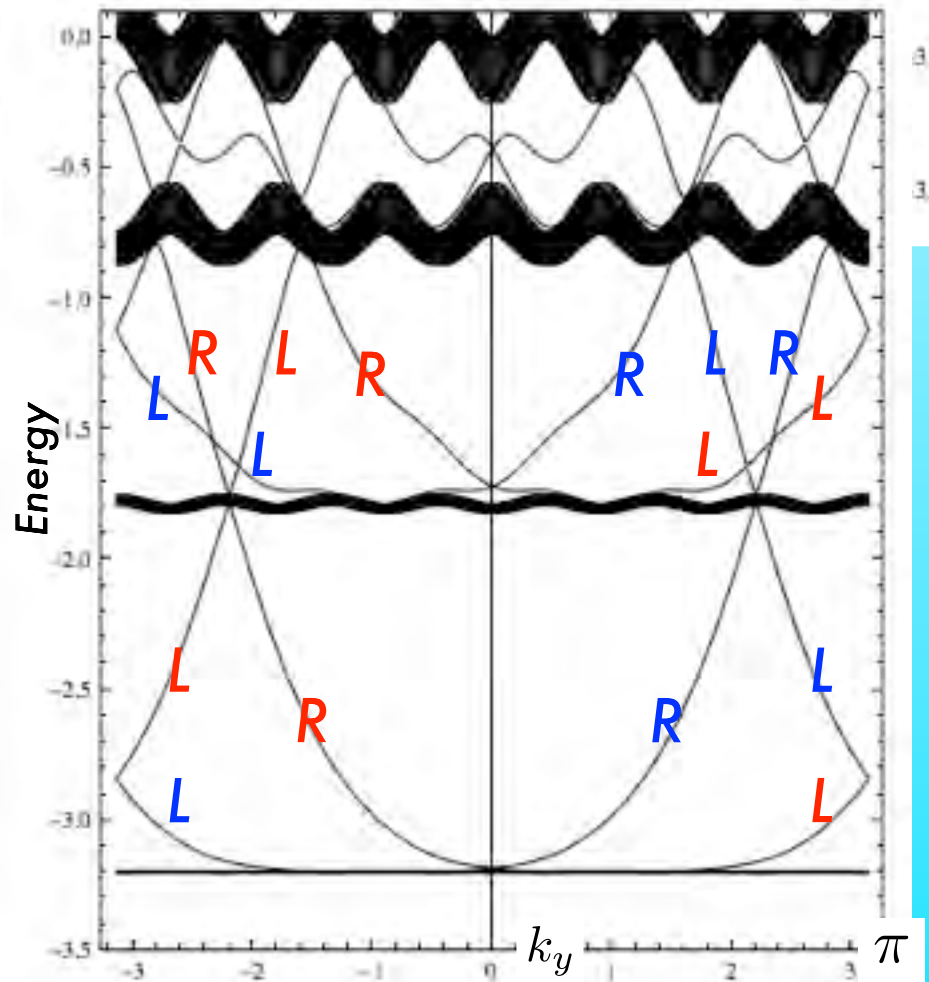
spin conserved case

decompose into up & down



down spin



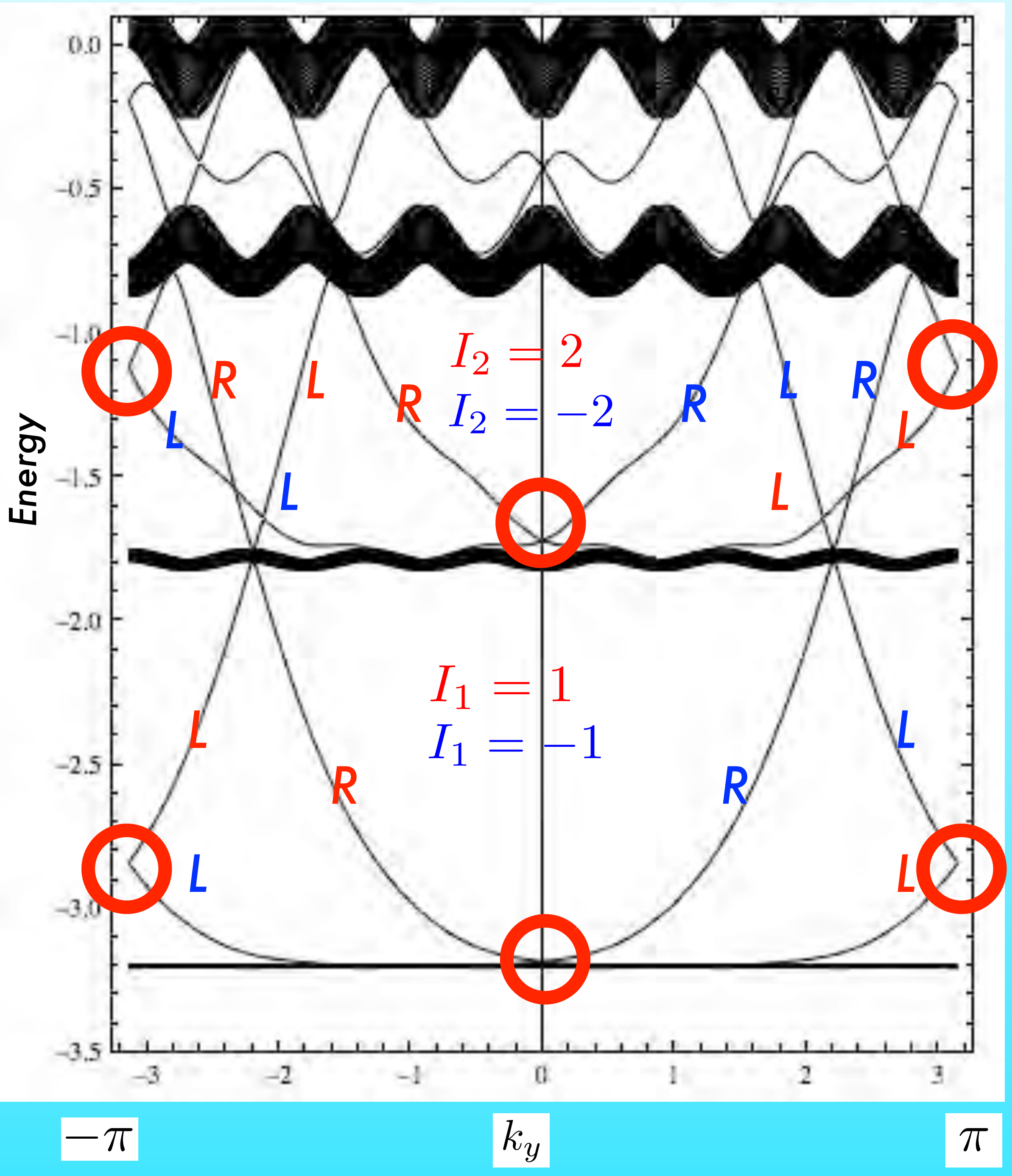


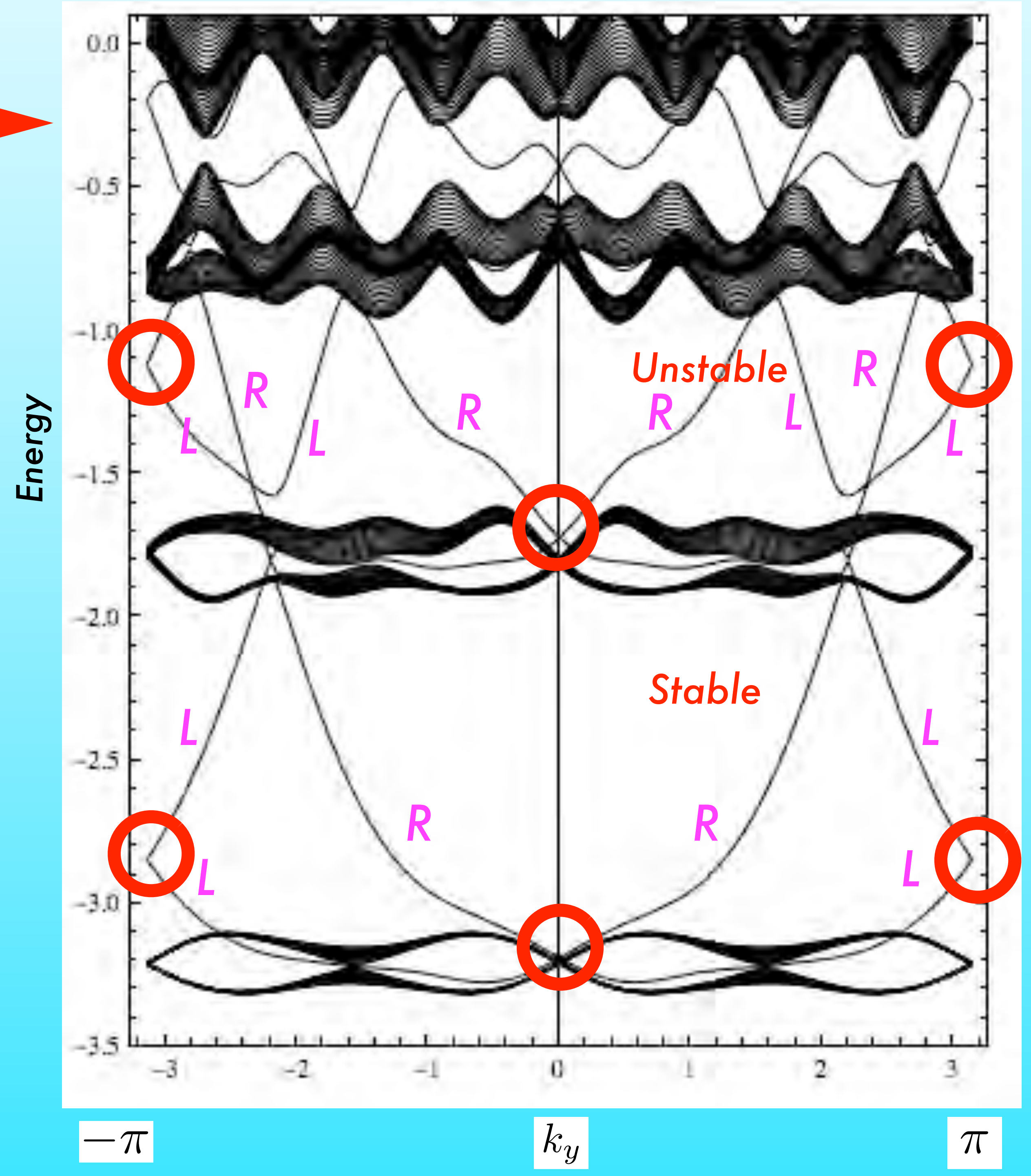
Identification of edge states (QSHE)

Spin Conserved

With Spin Orbit

Adiabatic modifications



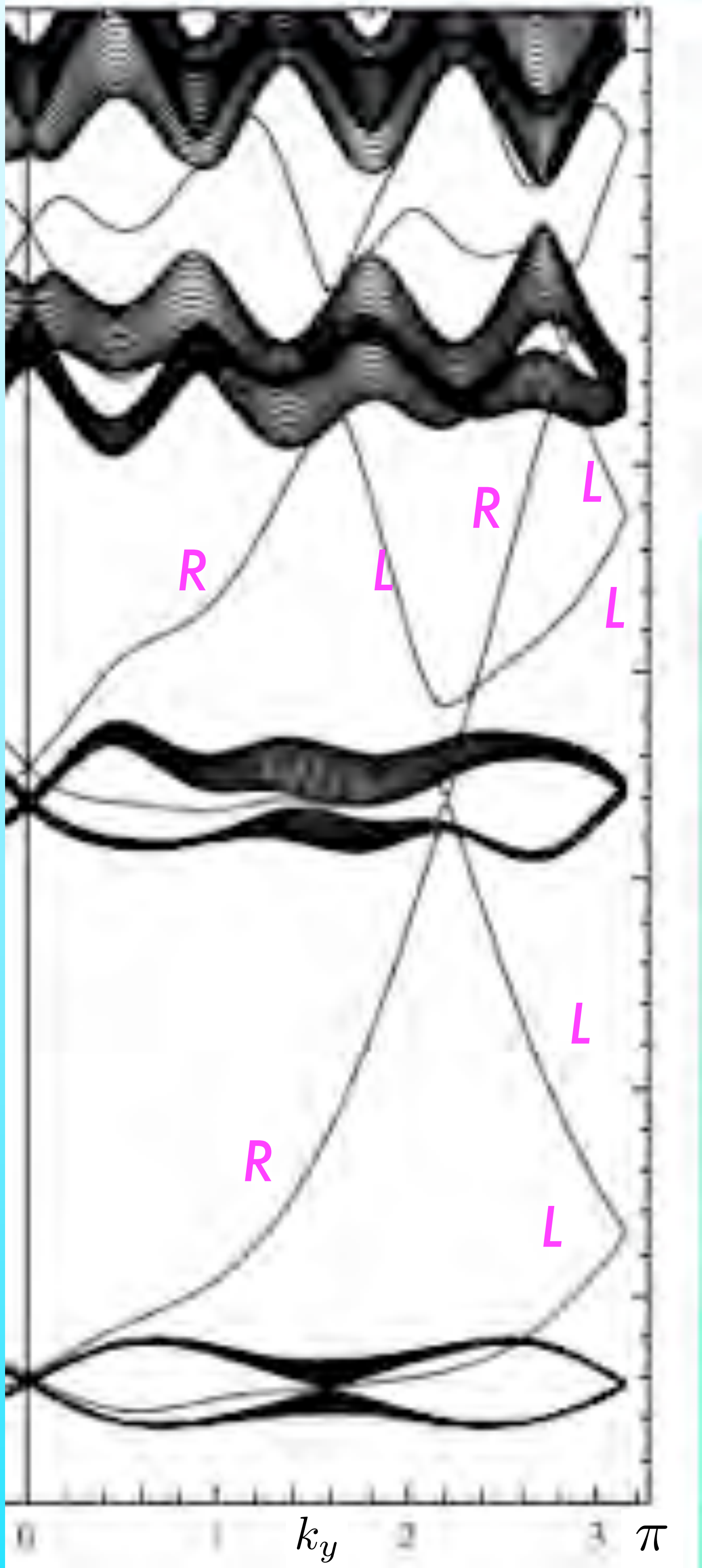


Kramers degeneracy at TR invariant momenta

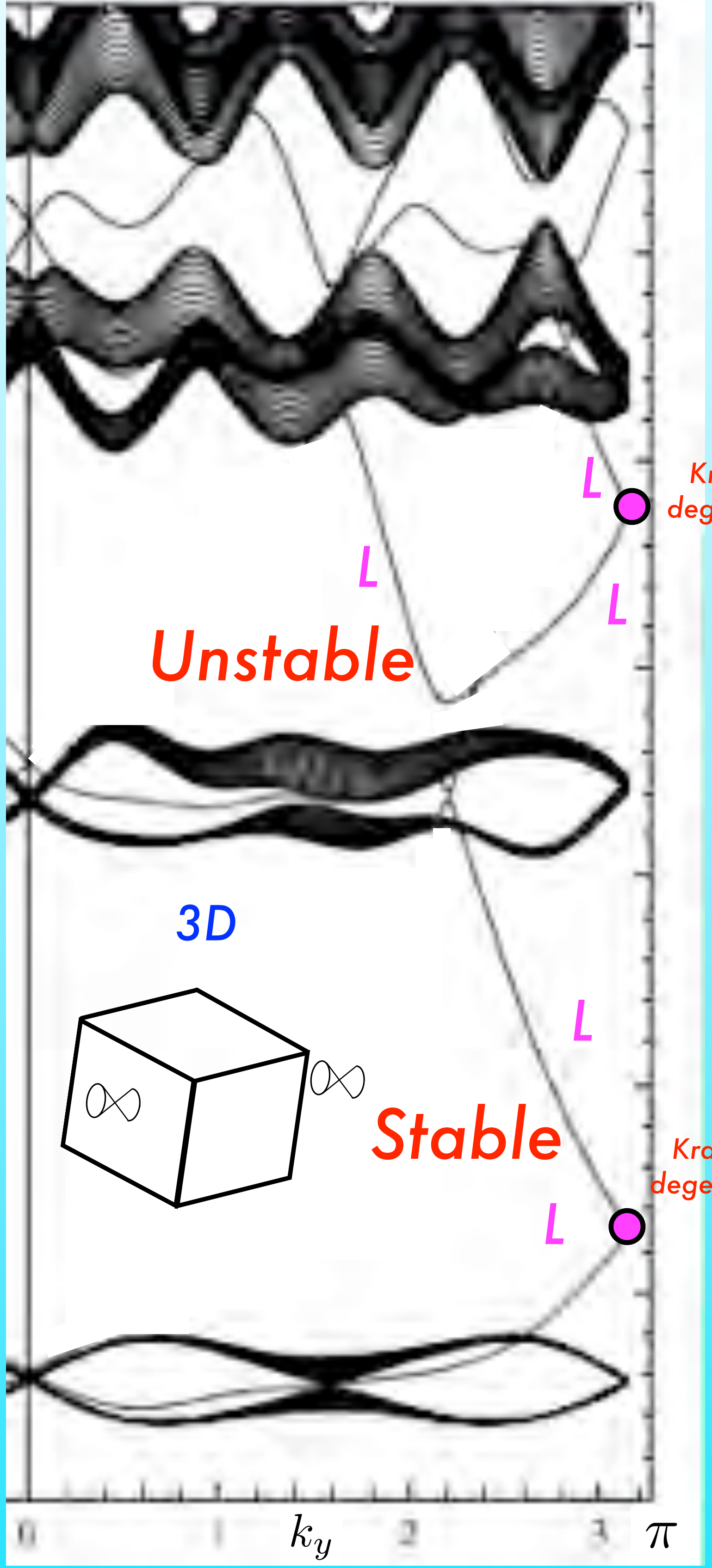
Adiabatic topological protection?

Identification of edge states (QSHE)

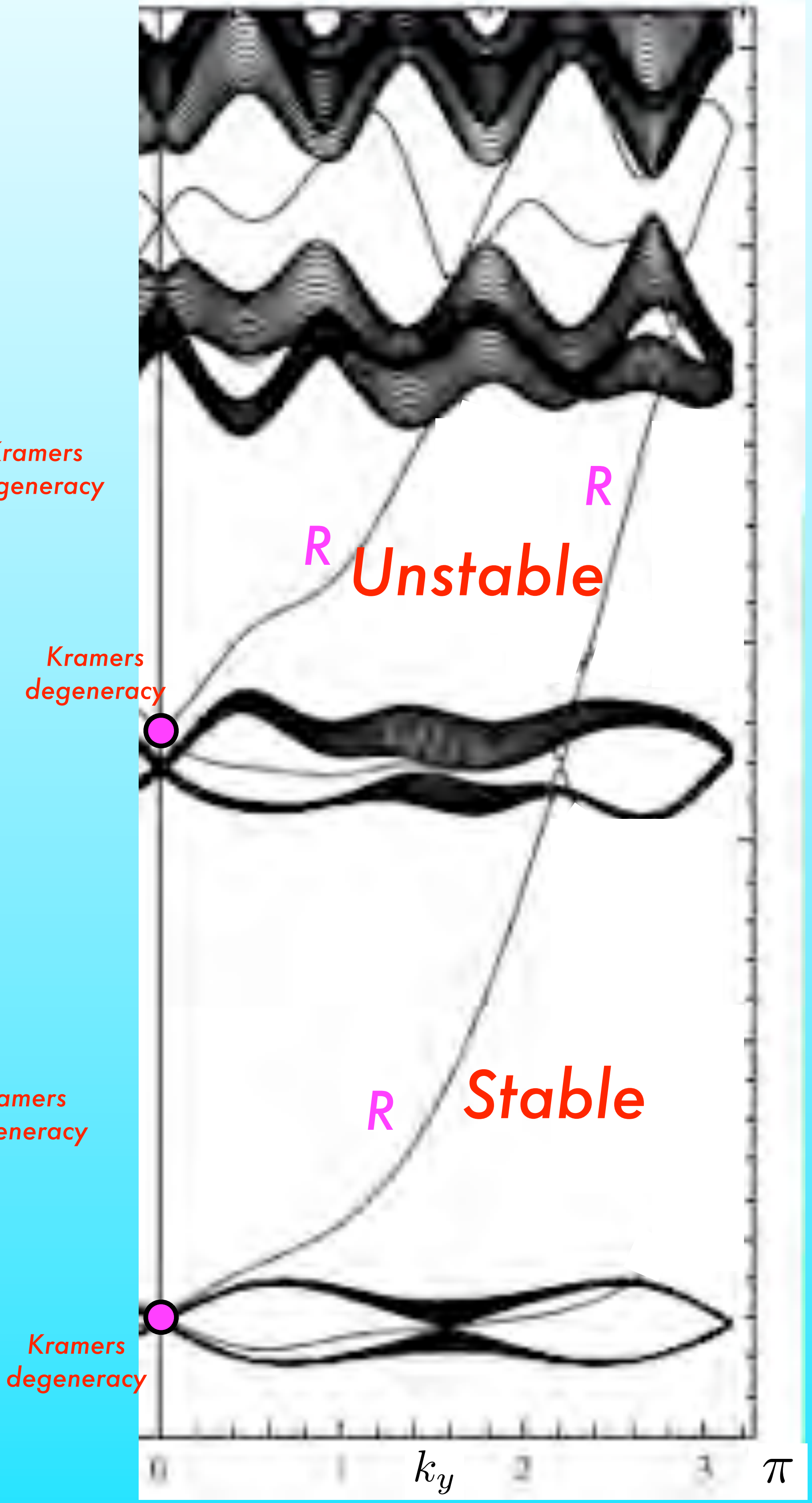
With Spin Orbit



L edge



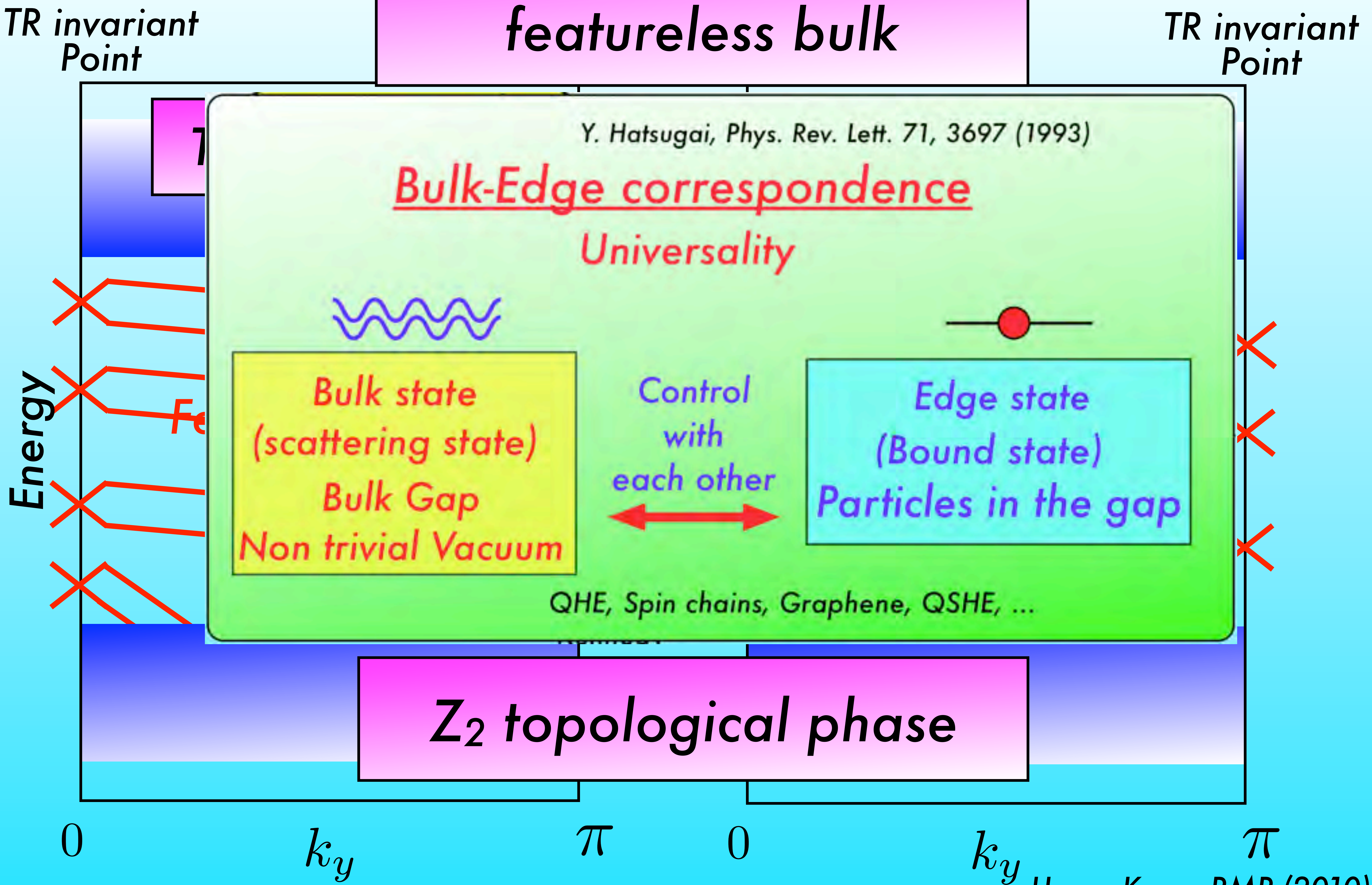
R edge



Z_2 edge states

Topolog states

Edges characterize the featureless bulk



Y. Hatsugai, Phys. Rev. Lett. 71, 3697 (1993)

Bulk-Edge correspondence

Universality

Bulk state
(scattering state)
Bulk Gap
Non trivial Vacuum

Control
with
each other

Edge state
(Bound state)
Particles in the gap

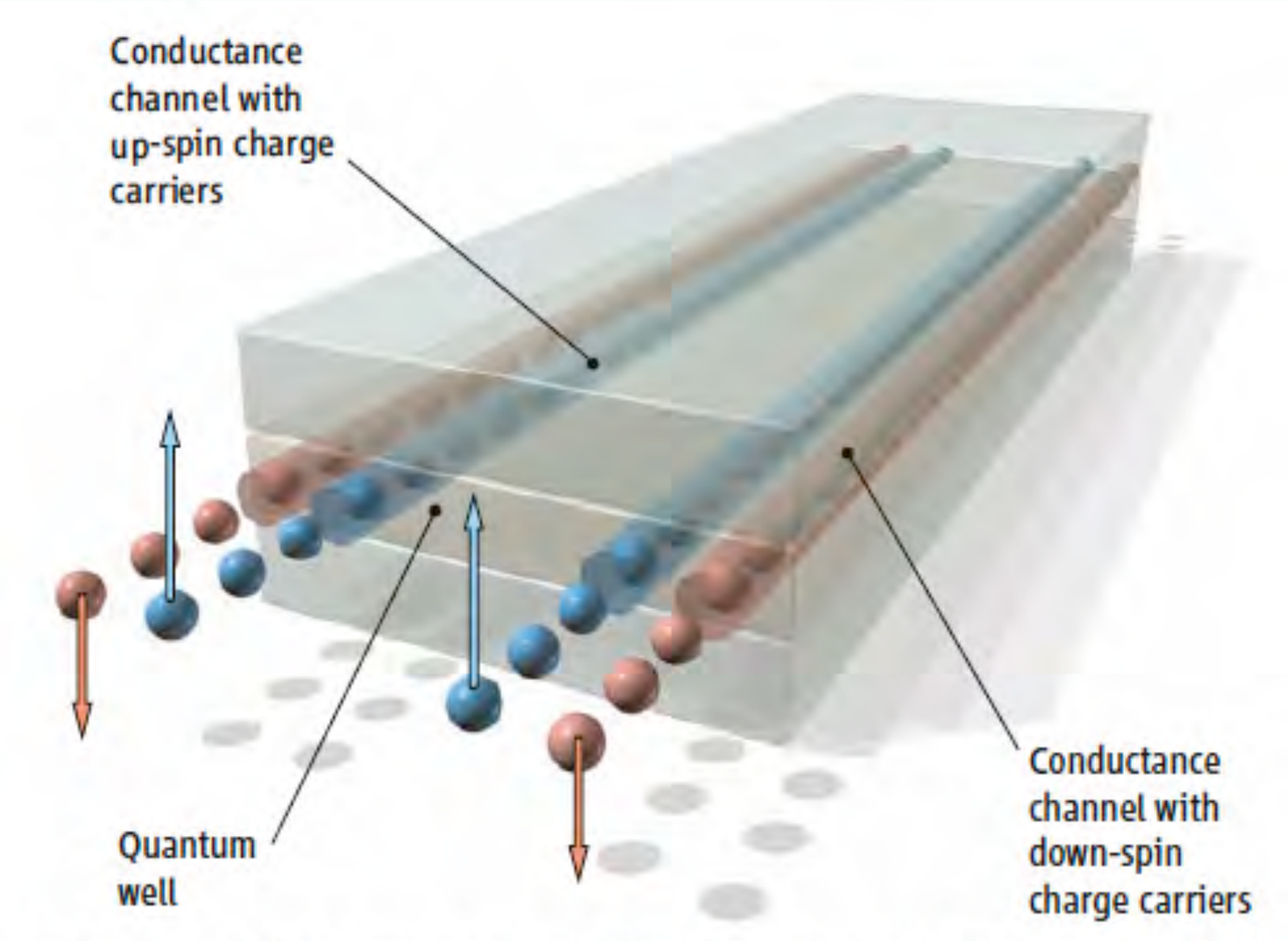
QHE, Spin chains, Graphene, QSHE, ...

Z_2 topological phase

Hasan-Kane, RMP (2010)

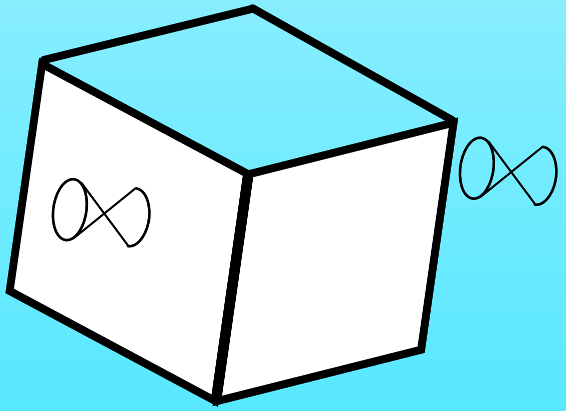
Spin Hall edge states

2D



Schematic of the spin-polarized edge channels in a quantum spin Hall insulator.

3D



....

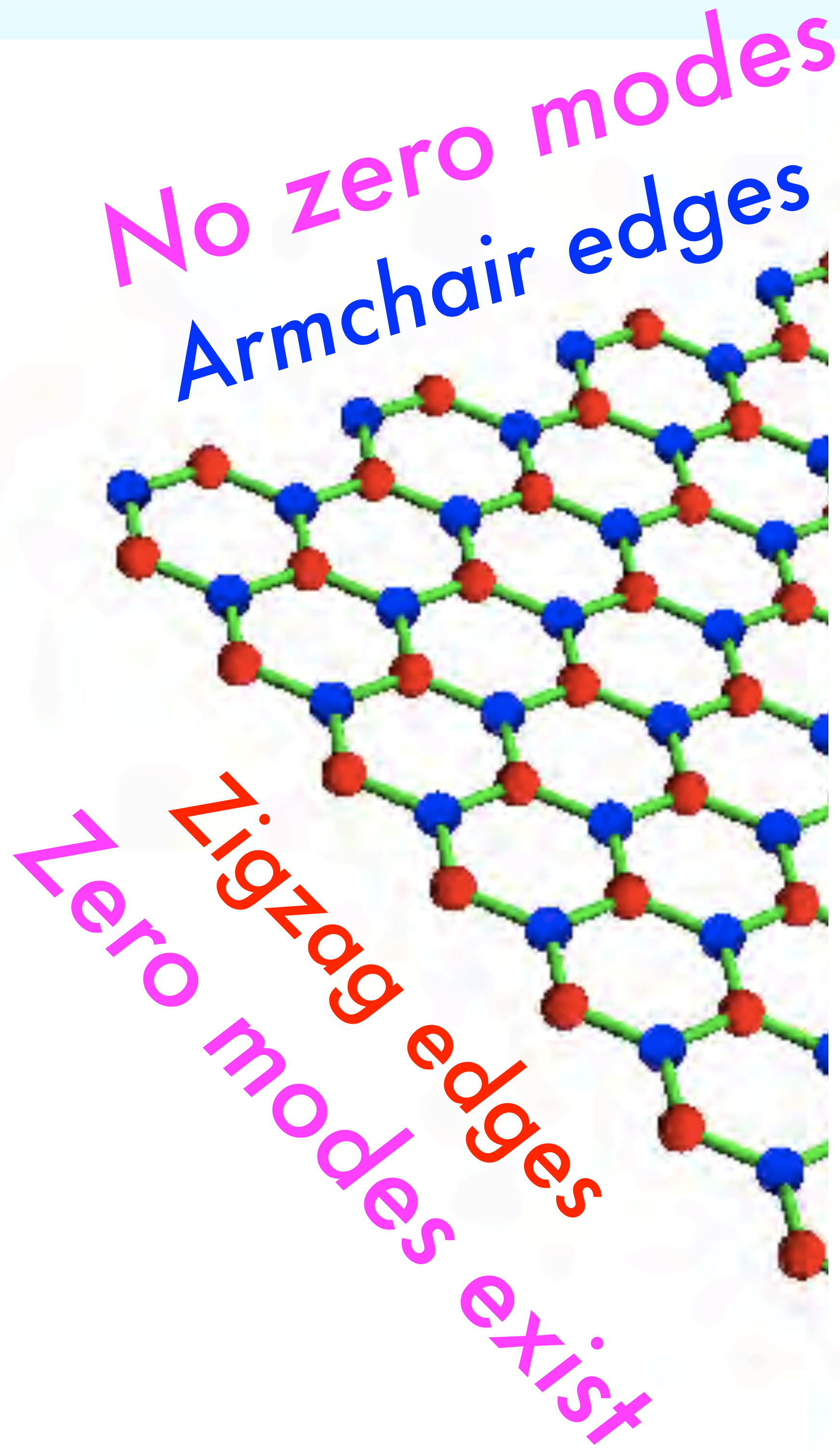
Konig, Wiedmann, Brüne, Roth, Hartmut Buhmann, Molenkamp,Qi and Zhang, Science 318, 776 (2007)

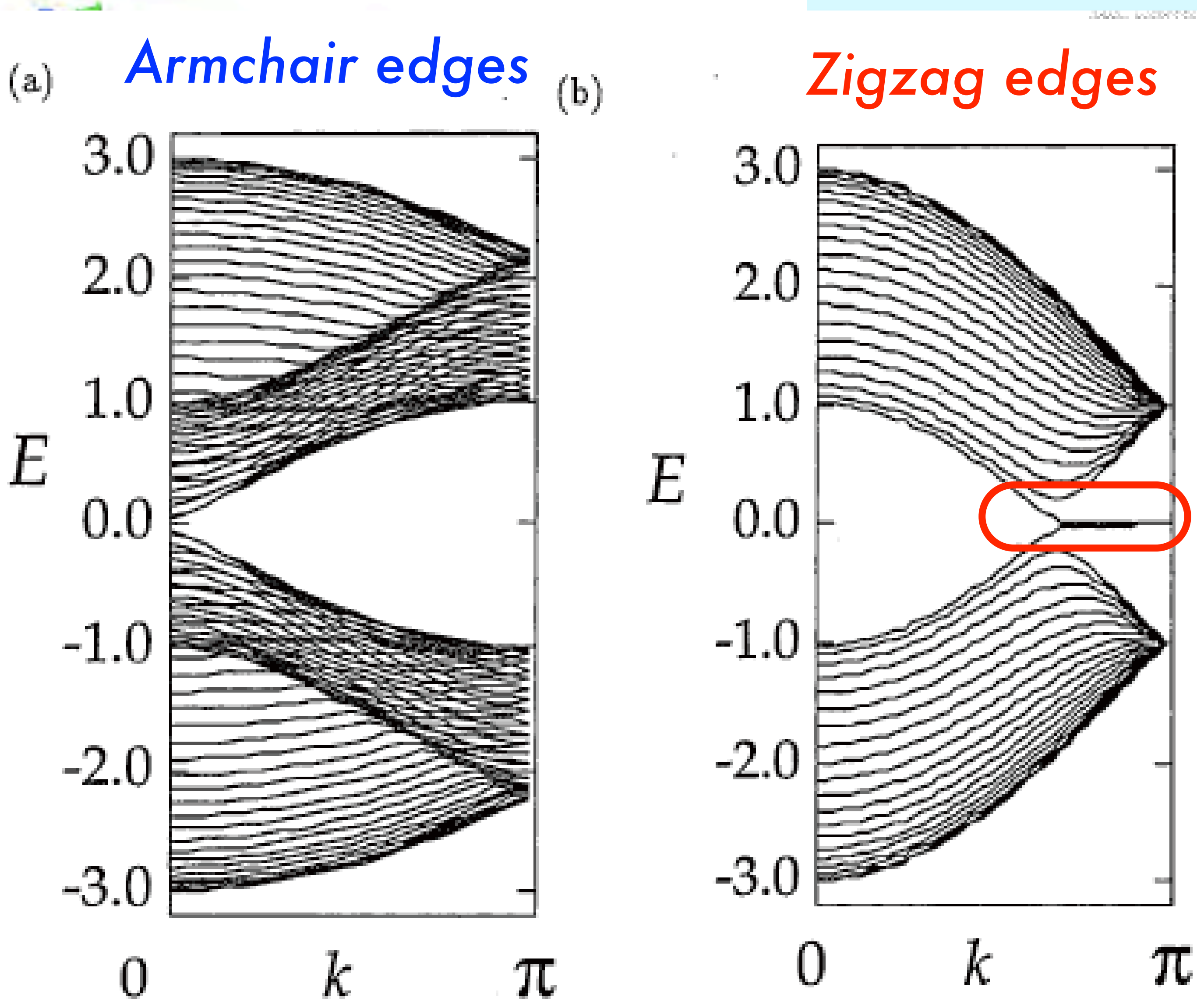
Another example: edge states

graphene

Z_2 edge states

Zero mode localized states ??



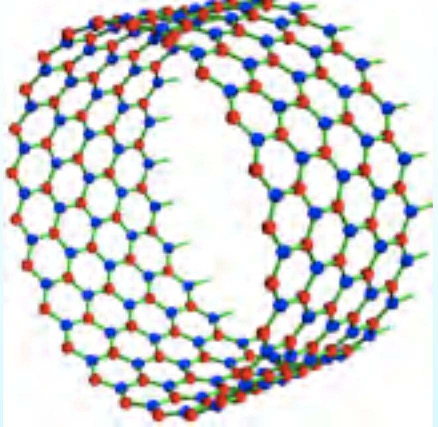


Fujita et al., J. Phys. Soc. Jpn. 65, 1920 (1996

Fujita et al.,'96

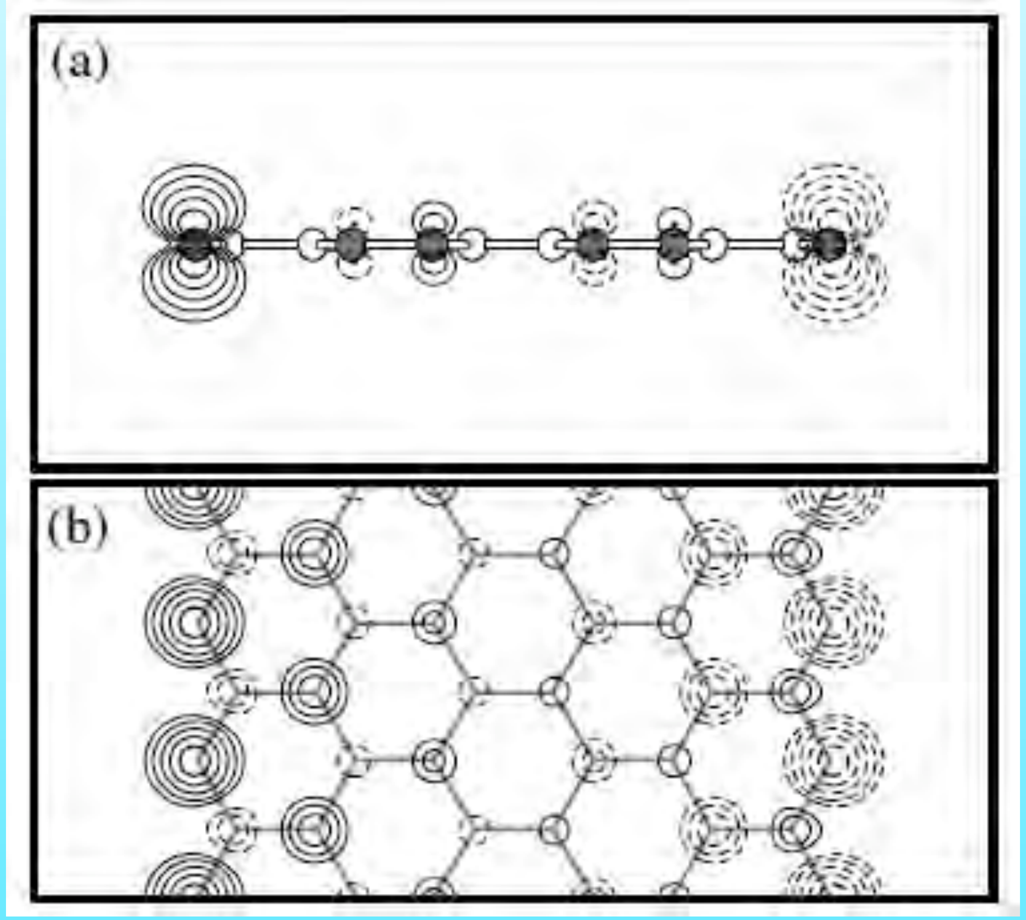
Another example: edge states

Z_2 edge states



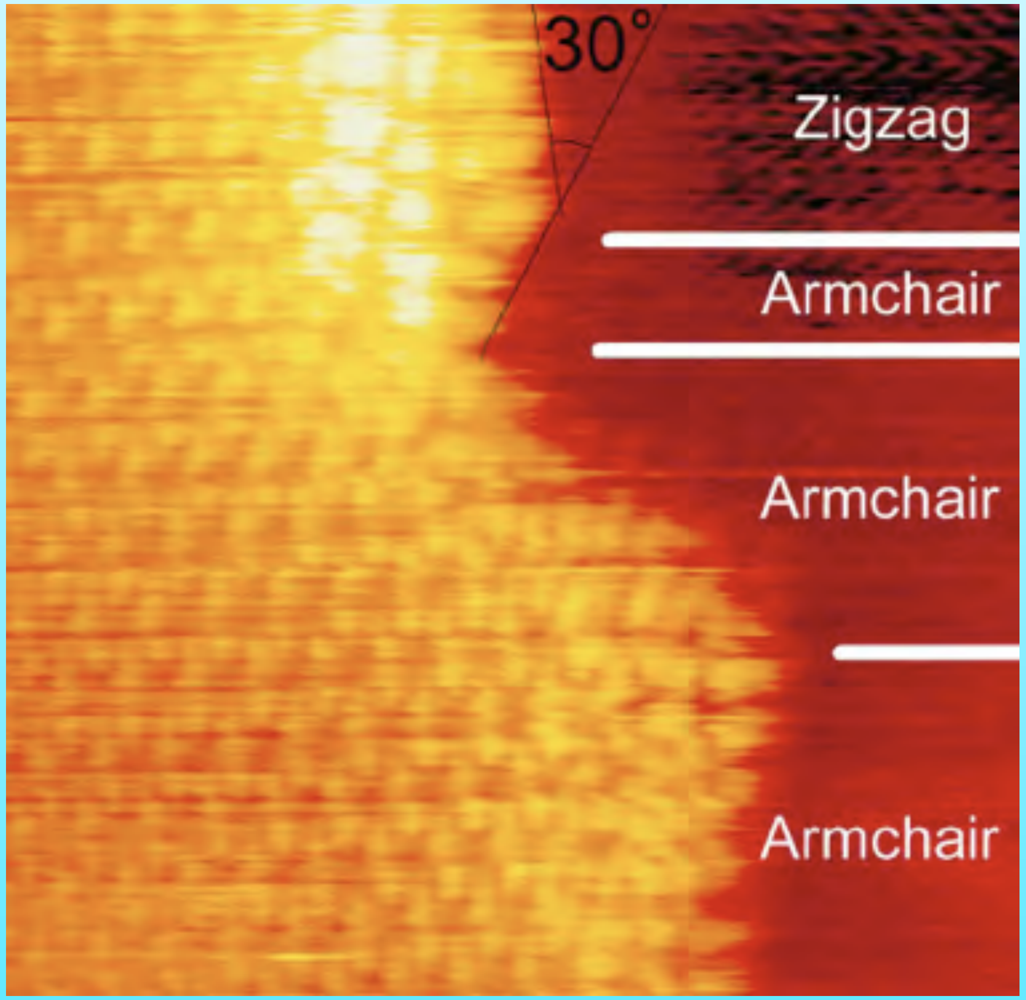
It's real !

First principle calculation



Okada and Oshiyama,
Phys. Rev. Lett. 87, 146803 (2001)

STM image

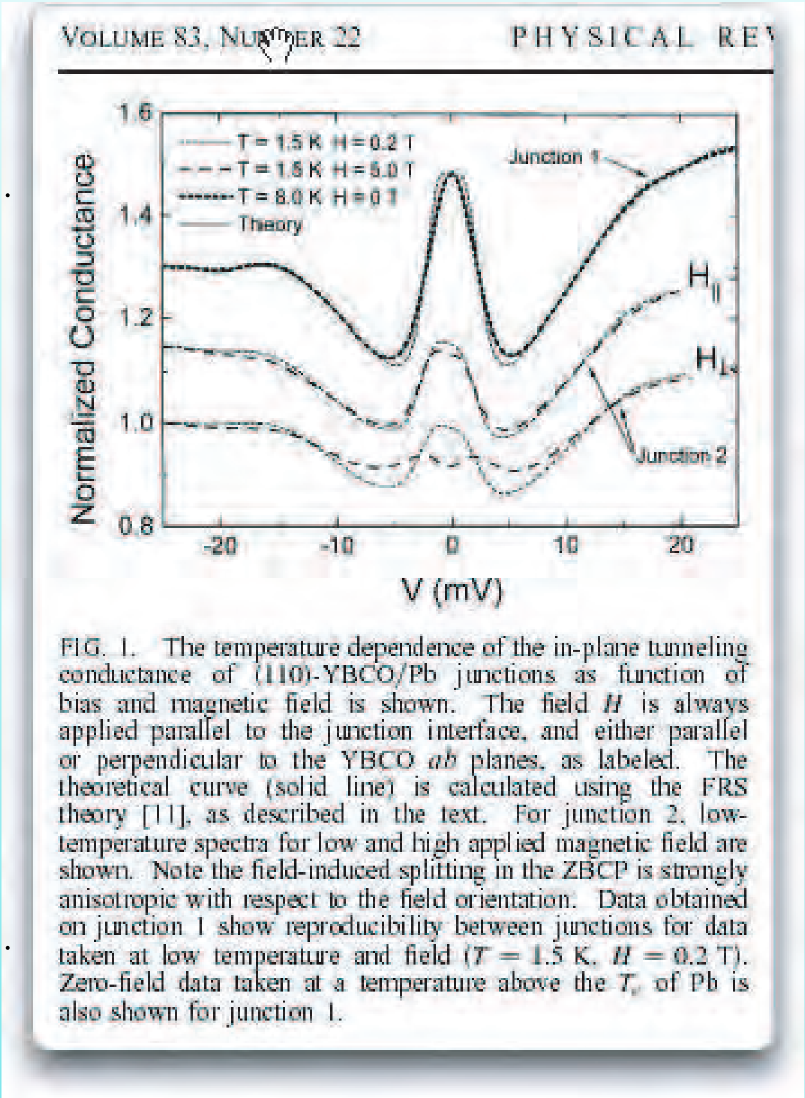


Kobayashi et al,
Phys. Rev. B71, 193406 (2005)

Another example: edge states

Zero Bias Conductance Peak in Anisotropic Superconductivity

d-wave superconductivity



VOLUME 83, NUMBER 22 PHYSICAL REV

FIG. 1. The temperature dependence of the in-plane tunneling conductance of (110)-YBCO/Pb junctions as function of bias and magnetic field is shown. The field H is always applied parallel to the junction interface, and either parallel or perpendicular to the YBCO ab planes, as labeled. The theoretical curve (solid line) is calculated using the FRS theory [11], as described in the text. For junction 2, low-temperature spectra for low and high applied magnetic field are shown. Note the field-induced splitting in the ZBCP is strongly anisotropic with respect to the field orientation. Data obtained on junction 1 show reproducibility between junctions for data taken at low temperature and field ($T = 1.5$ K, $H = 0.2$ T). Zero-field data taken at a temperature above the T_c of Pb is also shown for junction 1.

Zero Energy Boundary States of Anisotropic Superconductivity

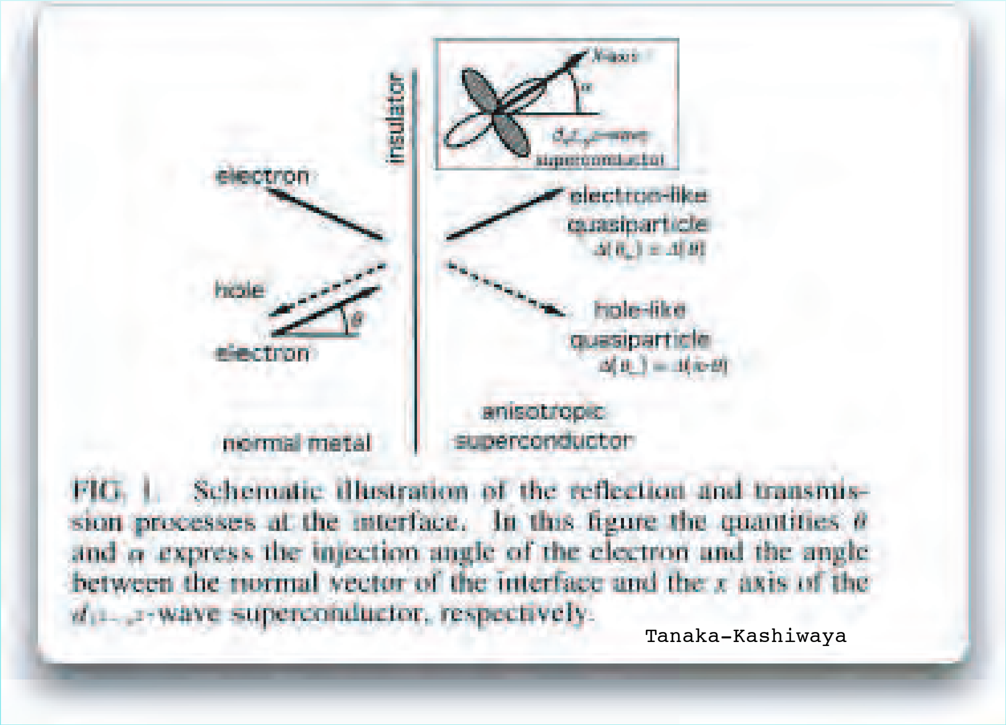


FIG. 1. Schematic illustration of the reflection and transmission processes at the interface. In this figure the quantities θ and α express the injection angle of the electron and the angle between the normal vector of the interface and the x axis of the $d_{x^2-y^2}$-wave superconductor, respectively.

Tanaka-Kashiwaya

L. J. Buchholtz,G. Zwicknagl, Phys. Rev. B 23, 5788 (1981) (p wave)

C.-R. Hu, Phys. Rev. Lett. 72, 1526 (1994) (d wave)

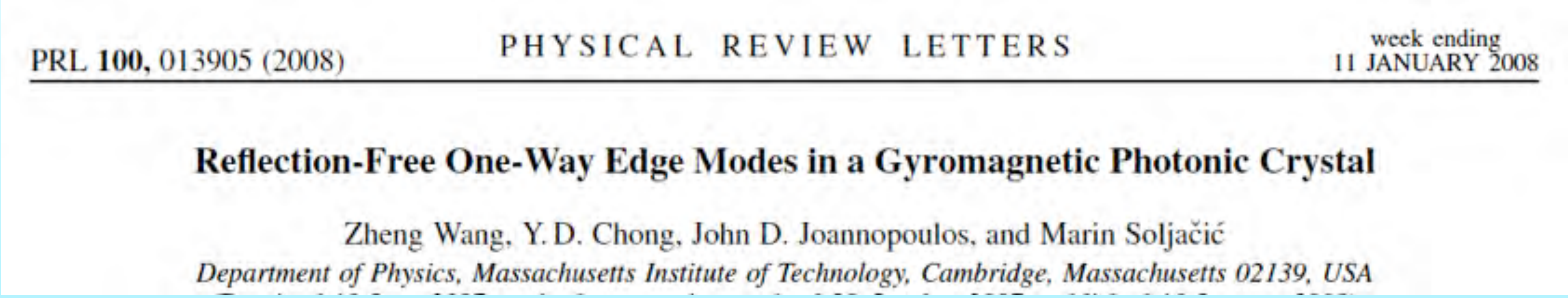
S. Kashiwaya, Y. Tanaka, Phys. Rev. Lett. 72, 1526 (1994)

M. Matsumoto and H. Shiba, JPSJ, 1703 (1995)

(fig.) M. Aprili, E. Badica, and L. H. Greene,Phys. Rev. Lett. 83, 4630 (1999)

Another example: edge states

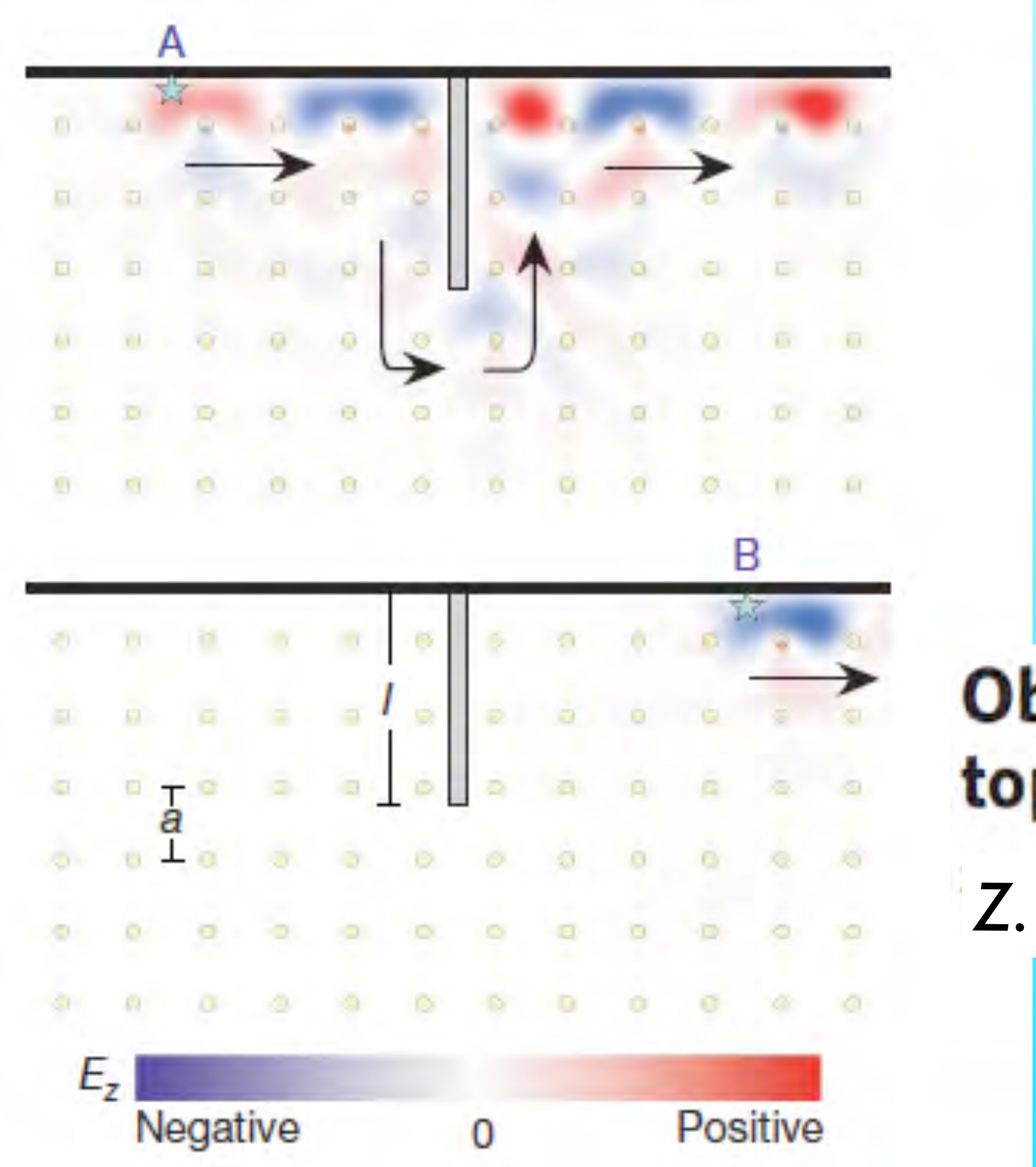
One way mode in photonic crystals

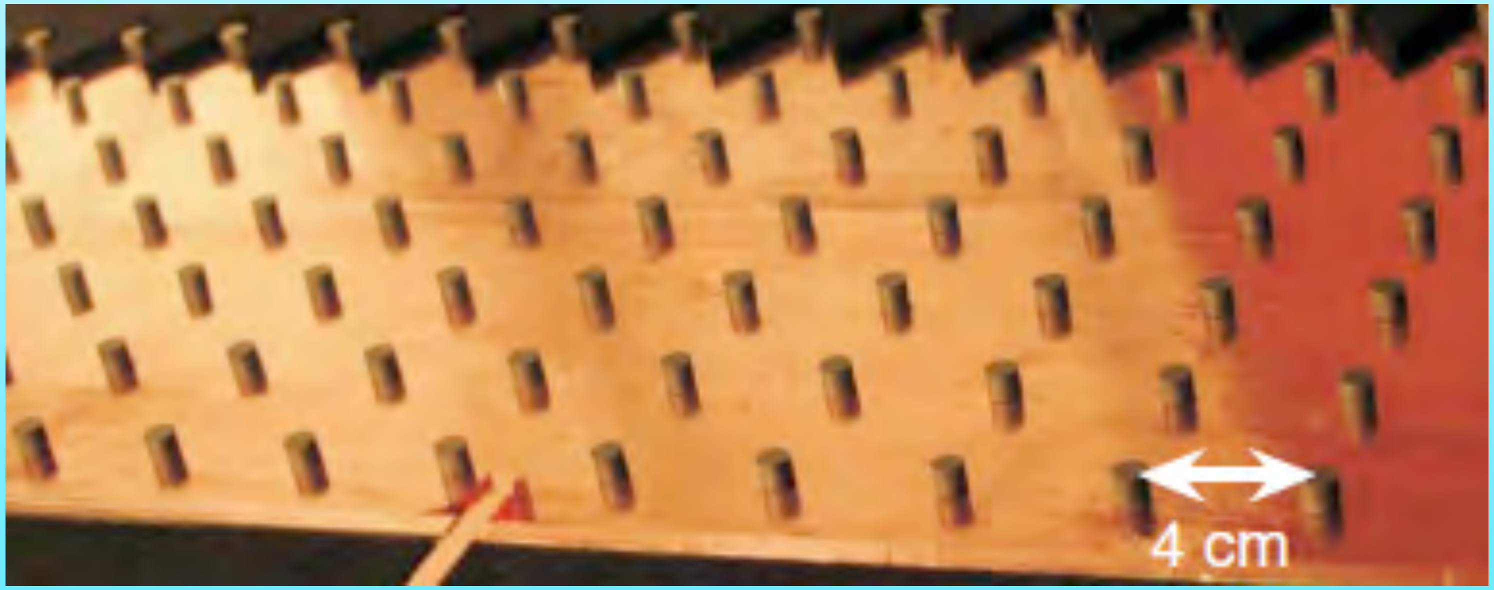


PRL **100,** 013905 (2008) PHYSICAL REVIEW LETTERS week ending 11 JANUARY 2008

Reflection-Free One-Way Edge Modes in a Gyromagnetic Photonic Crystal

Zheng Wang, Y. D. Chong, John D. Joannopoulos, and Marin Soljačić
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

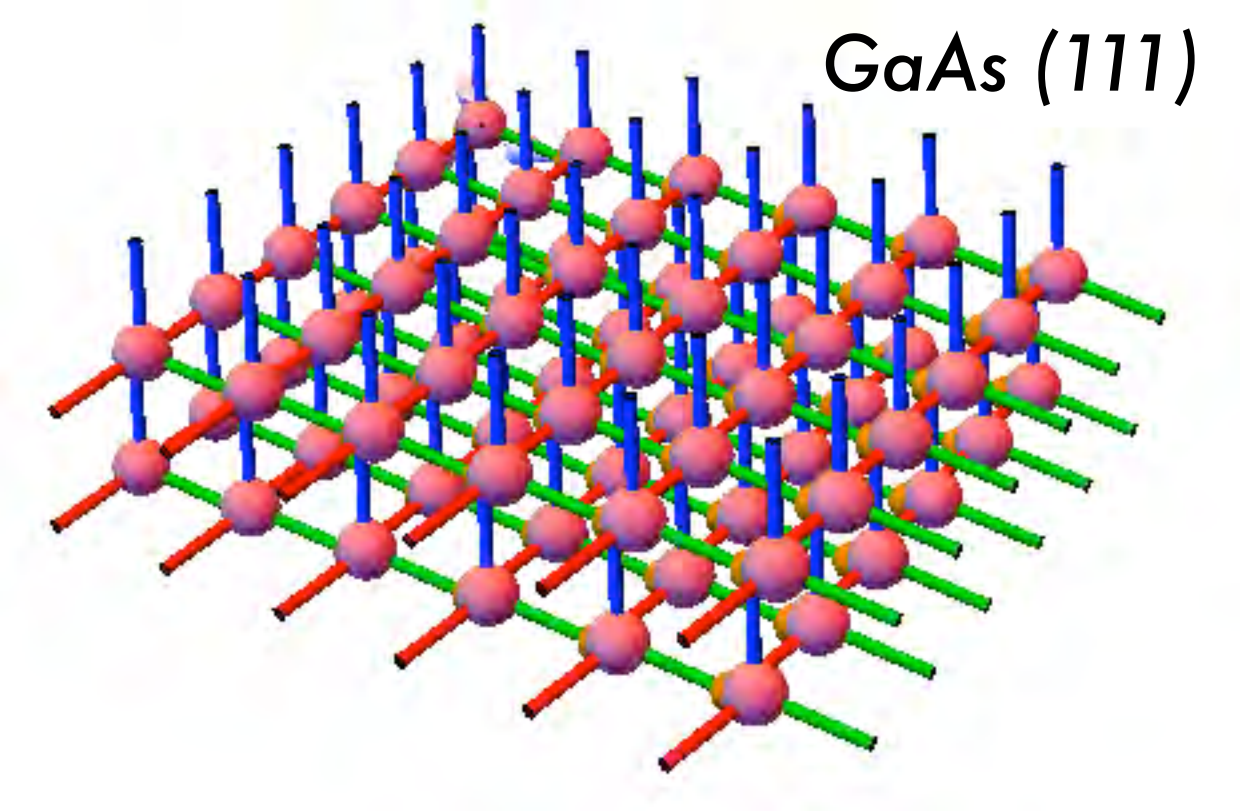


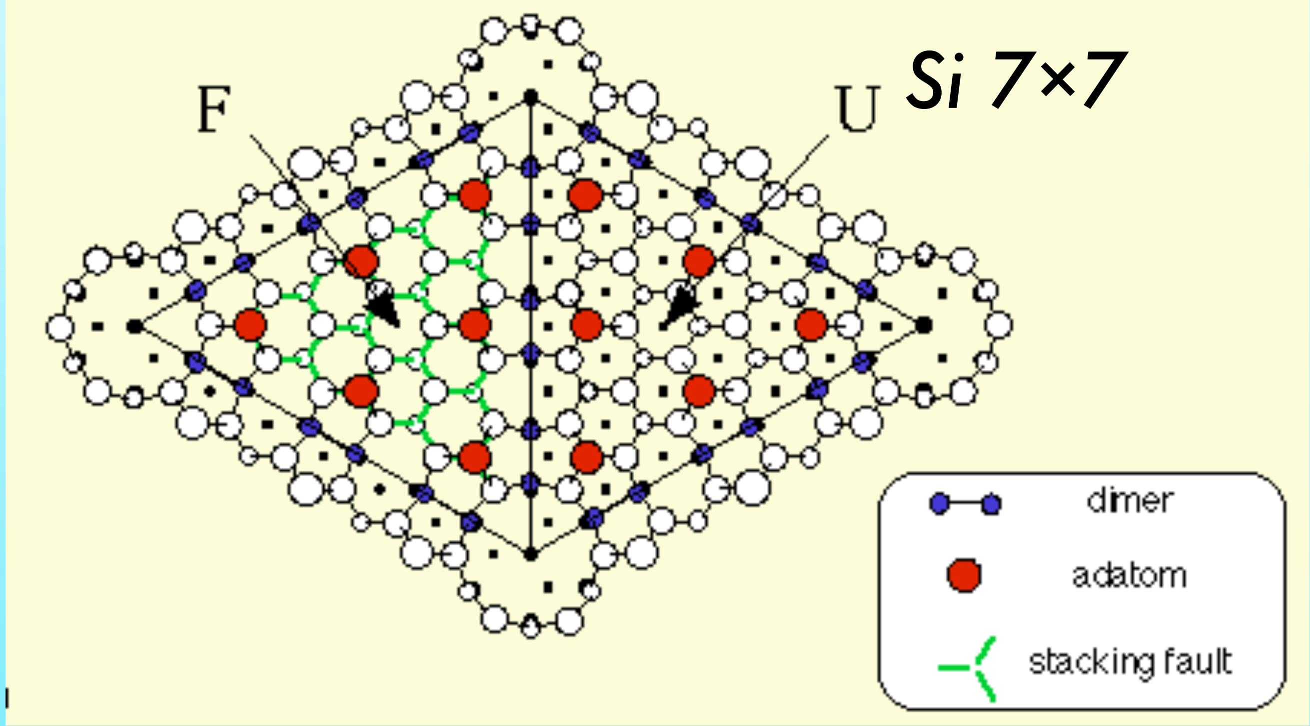


Observation of unidirectional backscattering-immune topological electromagnetic states

Z. Wang, Y. Chong, J.D.Joannopoulos, M. Solijacic
Nature 461 , 772 (2009)

Surface states of semiconductor without inversion symmetry





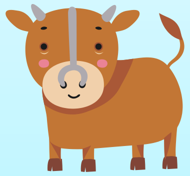
☑ Non topological generically
☑ Implicated by the bulk polarization

Edge States in Condensed Matter Physics

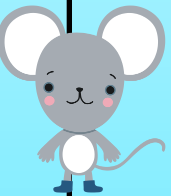
Z_2 edge states

Again its a zoo

- Bound states in quantum mechanics
  Levinson's theorem, Friedel sum rule
- Surface states in Semiconductors

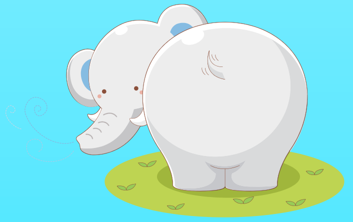


- Solitons in polyacetylene Su-Schriefer-Heeger '79
- Edge states in quantum Hall effects Halperin '82 Hatsugai '93
- Local moments in integer spin chains near the impurities
  Kennedy '90 Hagiwara-Katsumata-Affleck-Halperin-Renard '90



- Zero bias conductance peaks of the d-wave superconductors
  Hu, '94
- Zero energy localized states of graphene
  Fujita et al.'96 Ryu-Hatsugai'02
- Spin Hall Edge states Kane-Mele'95

more possibilities



- Edge states in 2D cold atoms optical lattices
  Scarola-Das Sarma., PRL 98, 210403 '07
- One-way edge modes in gyromagnetic photonic crystals
  Wang et al., PRL 100, 013905 '08

Bulk-Edge correspondence: Dirac fermions

2D Dirac fermions

Graphene d-wave superconductor surfaces of 3D TI ...

Edge states: quantum Hall edge state $B \neq 0$

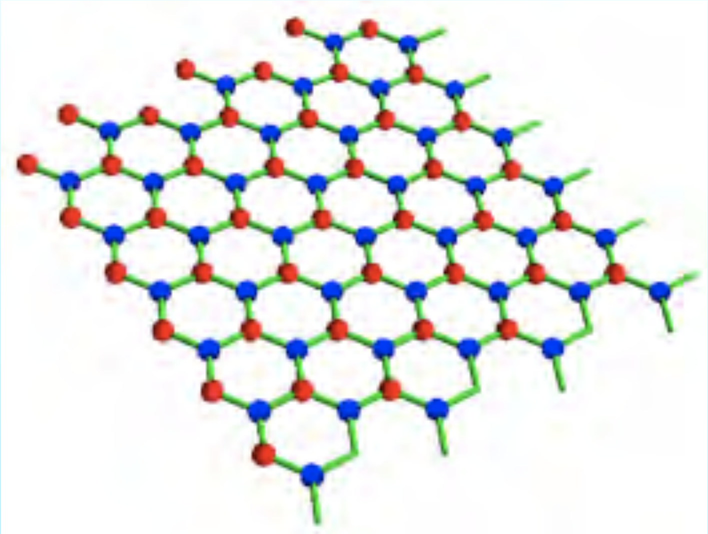
zero mode localized states $B = 0$

Universality in the zero modes
of Dirac Fermions

1D Dirac fermions :

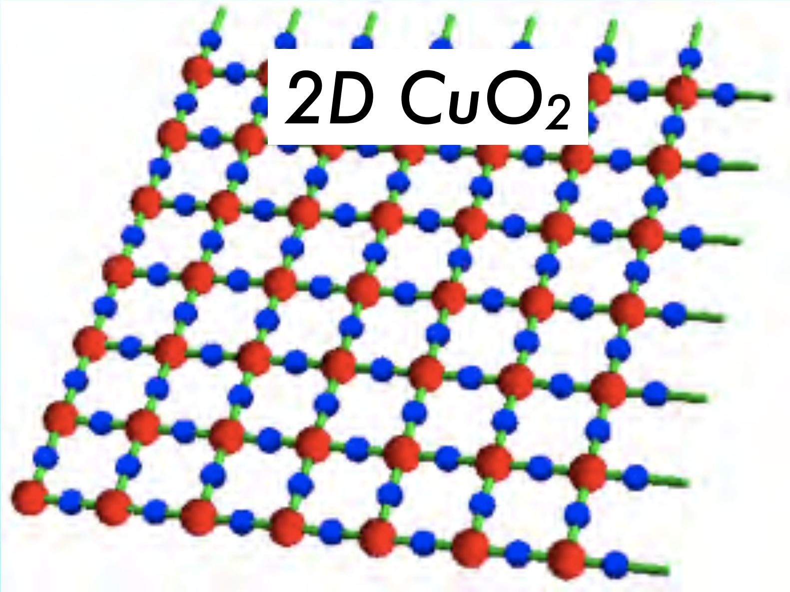
Su-Schriefer-Heeger '79
Witten '81

Universality in the zero modes of Dirac Fermions



2D Dirac fermions :

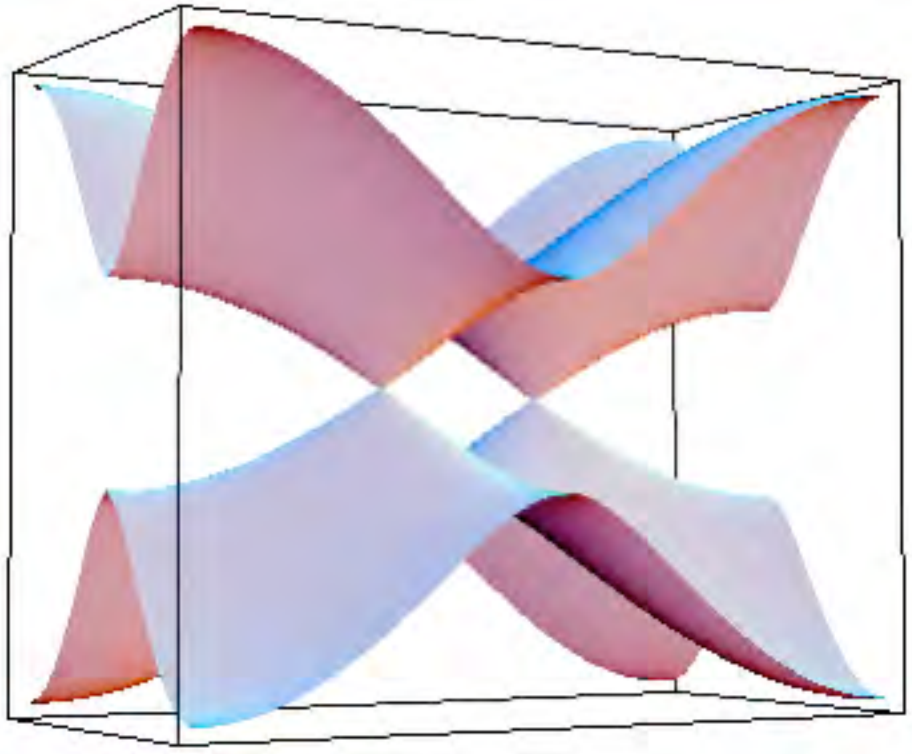
Edge States



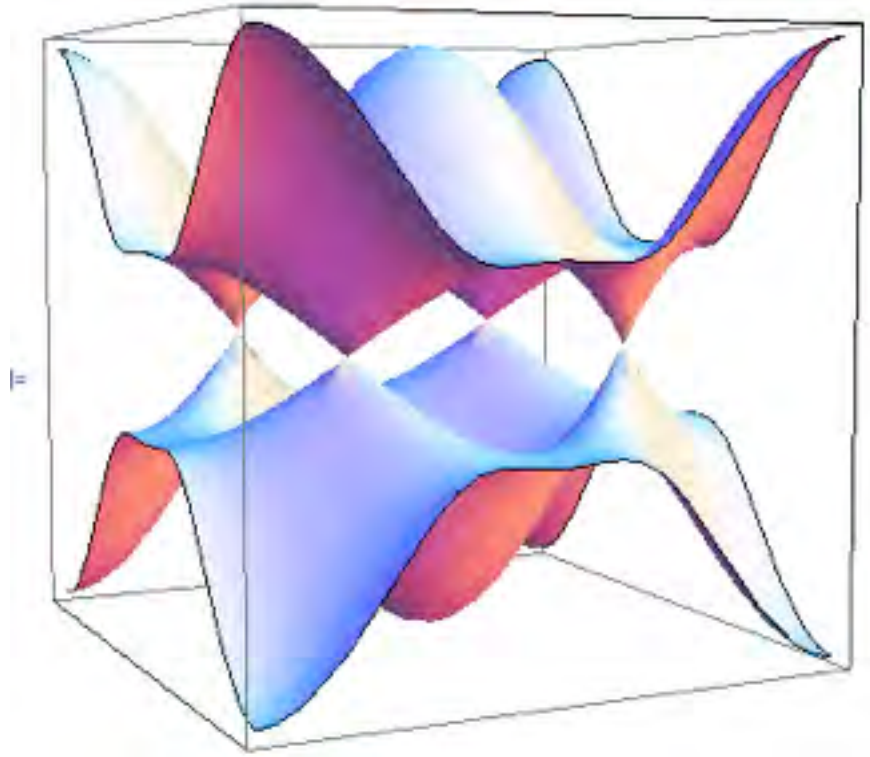
Zero mode localized states ??

YH, '09 (review)

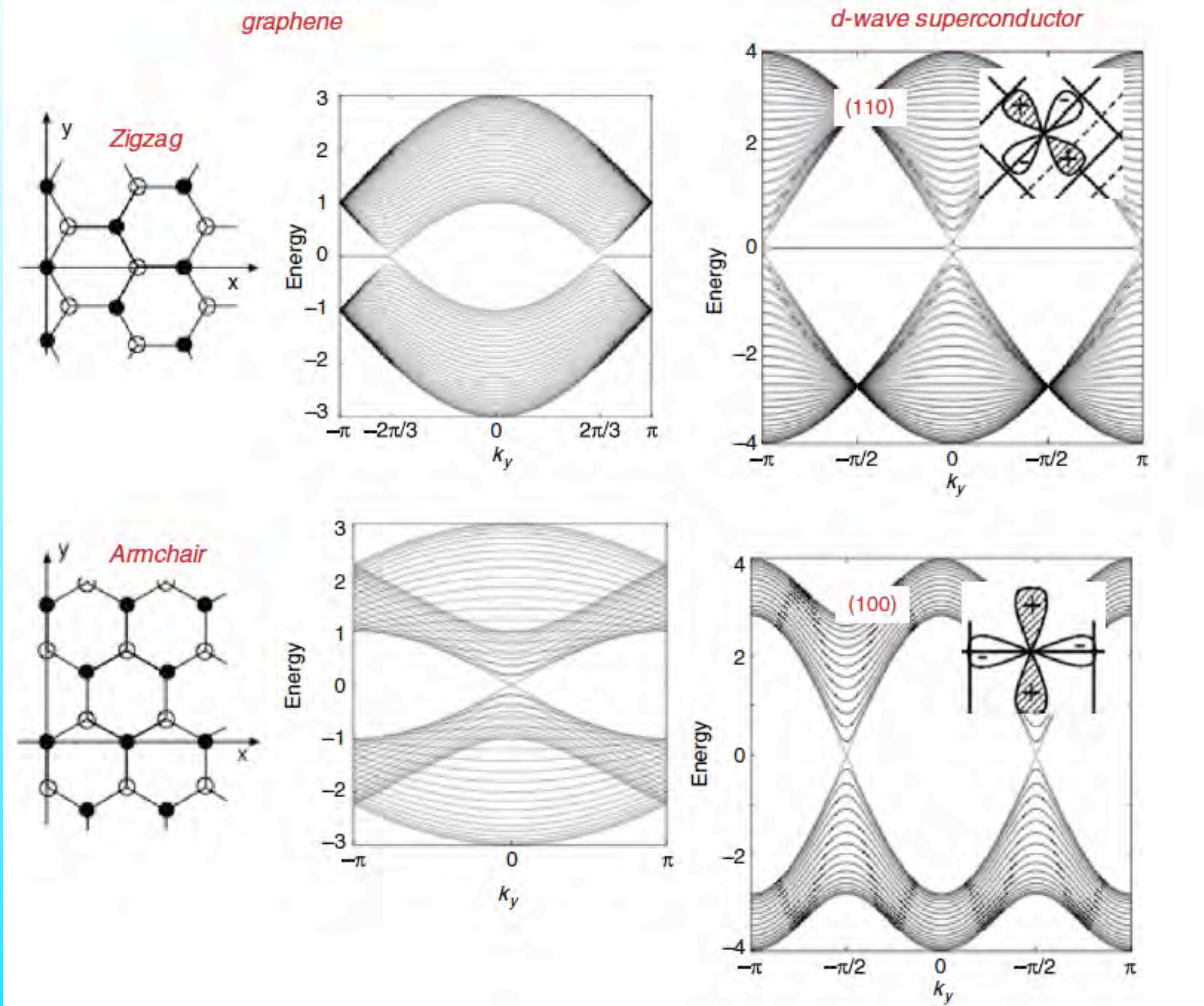
Graphene



d-wave superconductor



Analogy between graphene & d-wave superconductor



Universality of Zero Energy Edge States

'02–'04 S. Ryu & YH

Boundary magnetic moments

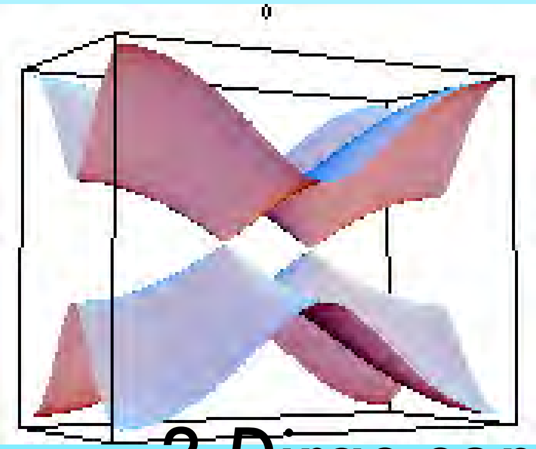
Spontaneous breaking of these chiral symmetries
: Peierls instabilities of Flat (edge) bands

Spontaneous local flux generation near defects

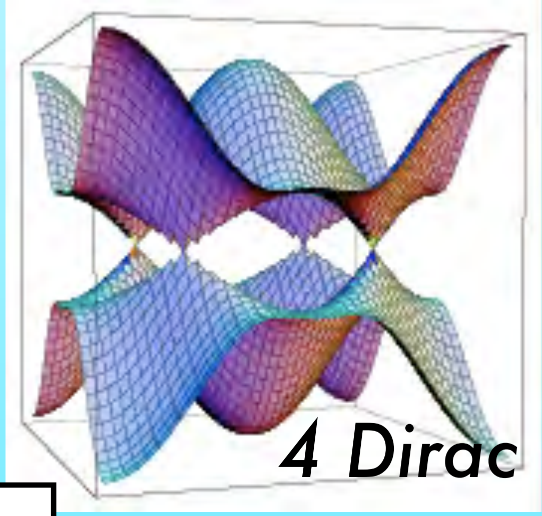
Zero bia

graphene

d-wave superconductor


2 Dirac cones

These 2 systems are topologically equivalent


4 Dirac cones

Symmetry protected
Zero modes of Dirac fermions
:1D Flat Band of edge states

Γ :Bipartite

(A-B sublattice symmetry)

Γ:Time Reversal

(Real Order parameter)

$^\exists\Gamma$ chiral symmetry

$$\{\Gamma, H\} = \Gamma H + H\Gamma = 0\ ,\ \Gamma^2 = 1$$

When the zero modes exist ?

Lattice analogue of
Witten's SUSY QM

S.Ryu & Y.Hatsugai, Phys. Rev. Lett. 89, 077002 (2002)
Y.Hatsugai., J. Phys. Soc. Jpn. 75 123601 (2006)
Kuge, Maruyama, Y. Hatsugai, arXiv:0802.2425

Edge states with boundaries
Determined by the Berry phase of the bulk (without boundaries)

Zak $\gamma = \int A = \int d\vec{k} \cdot \vec{A}$ $\vec{A} = \langle \psi(k) | \vec{\nabla}_k \psi(k) \rangle$

Require Local Chiral Symmetry
(ex. bipartite)

$$\{\Gamma, H\} = \Gamma H + H \Gamma = 0$$

$\Rightarrow$

Quantized

$$\gamma = \int A = \begin{cases} \pi \\ 0 \end{cases}$$

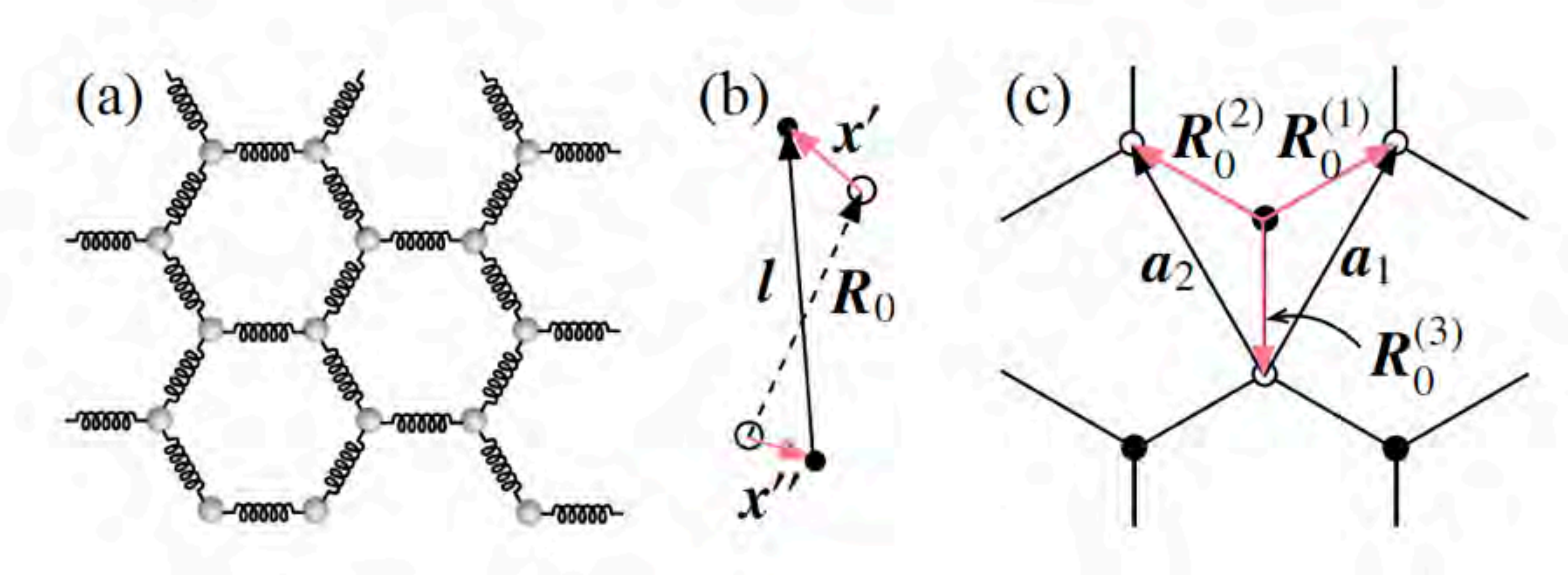
$\boxed{\gamma = \pi}$ $\Rightarrow$ Zero energy localized states EXIST

: There exists odd number of zero modes

Bulk-edge correspondence: "Bulk determines the edges"

Manipulation of Dirac Cones in Mechanical Graphene

T. Kariyado & Y.Hatsugai, arXiv:1505.06679



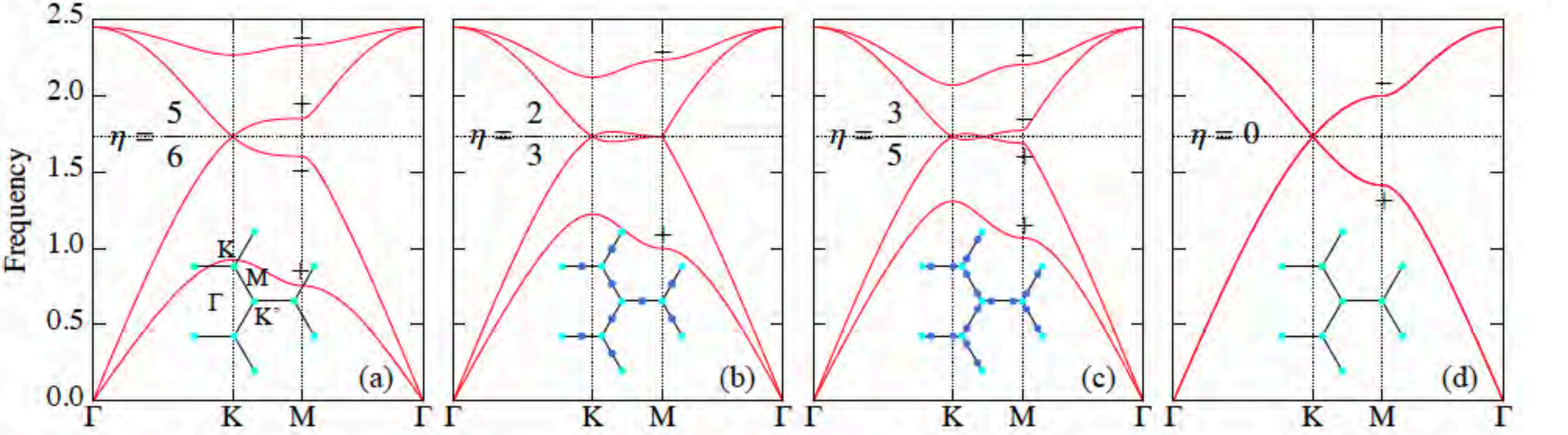
Spring-mass model on honeycomb lattice

Classical system governed by Newton law

$$U_s = \frac{1}{2}\kappa(l - l_0)^2$$

Dirac Cones in phonon spectrum

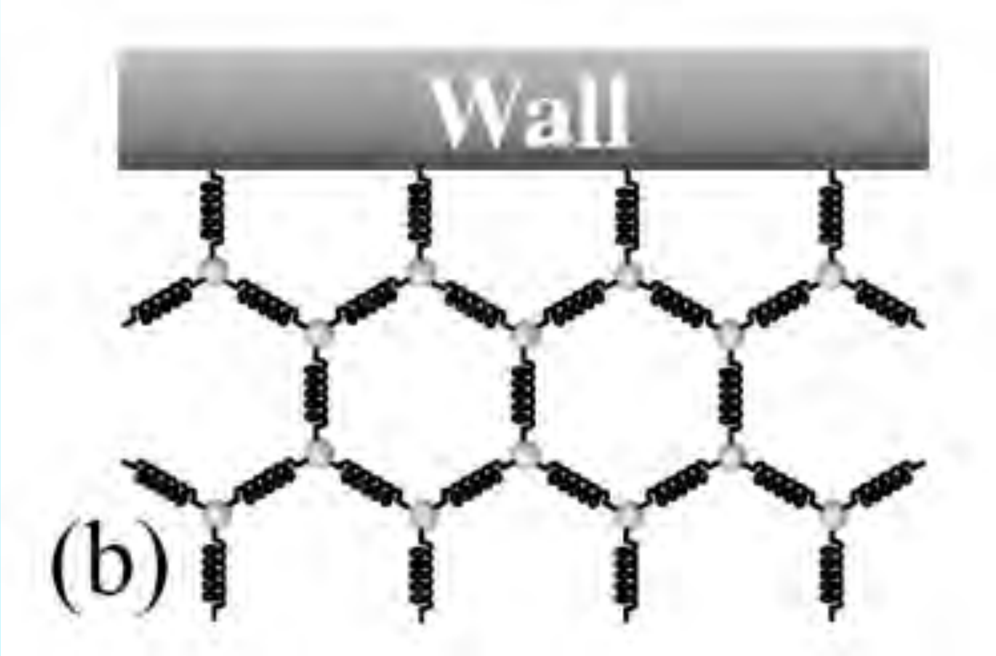
η : controled by spring tension

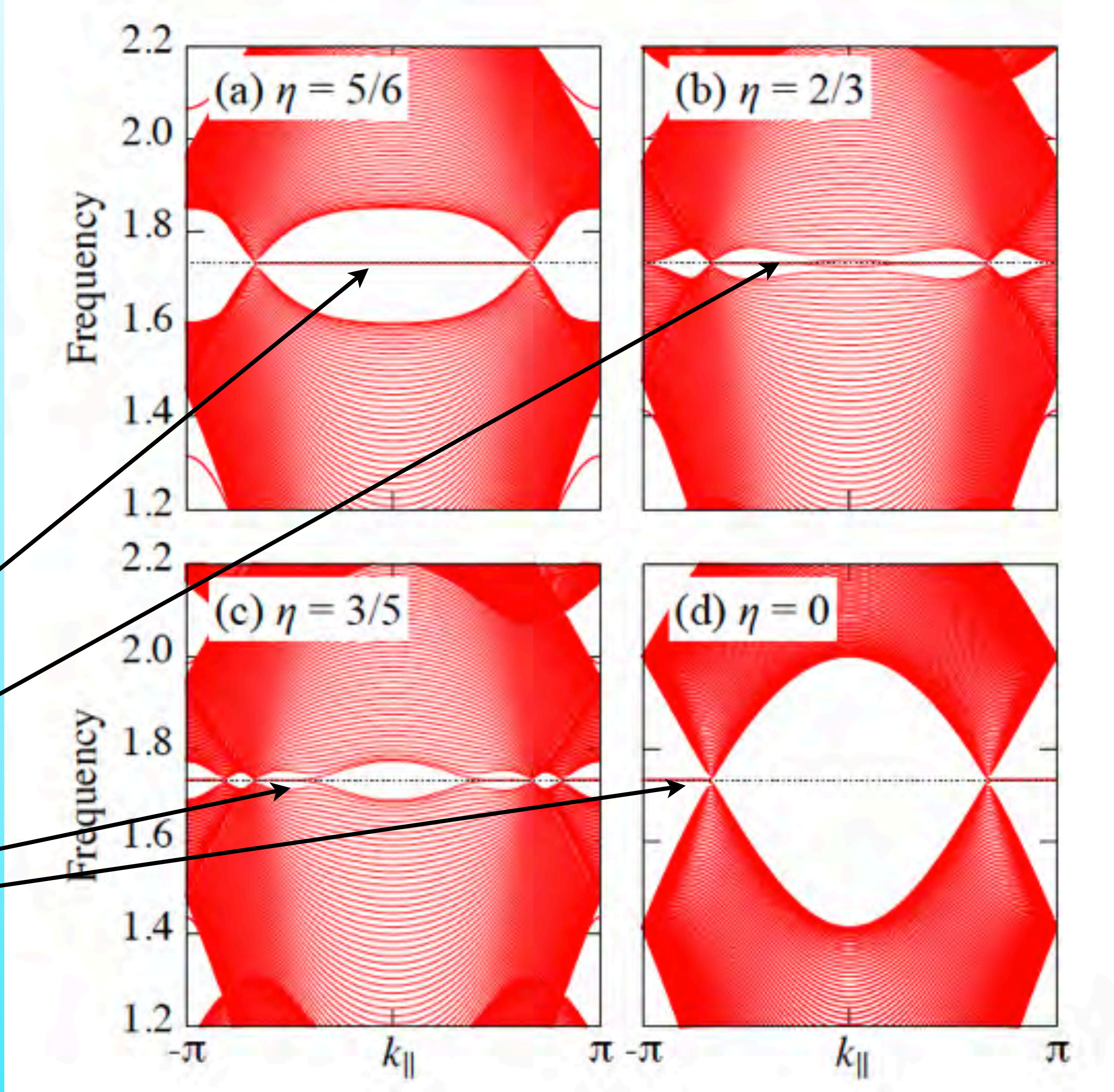


Dirac cones appear and manipulated by spring tension

Phonon spectrum on cylinder

η : controled by spring tension



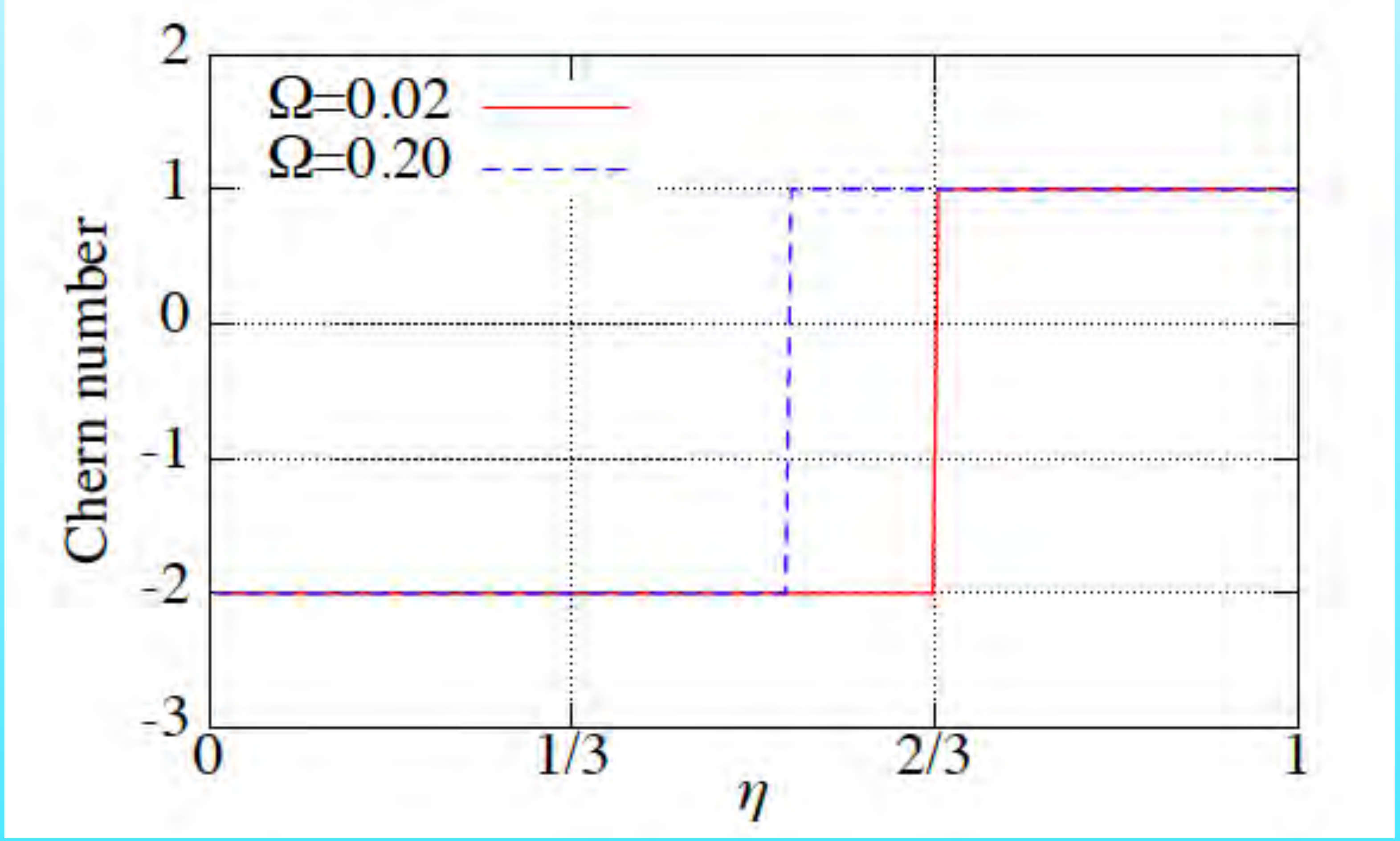


Edge states

Dirac cones appear and manipulated by spring tension

With rotation: time-reversal symmetry breaking Coriolis force (effective magnetic field)

Topological change associated with Chern number jumps



Time evolutions in real space

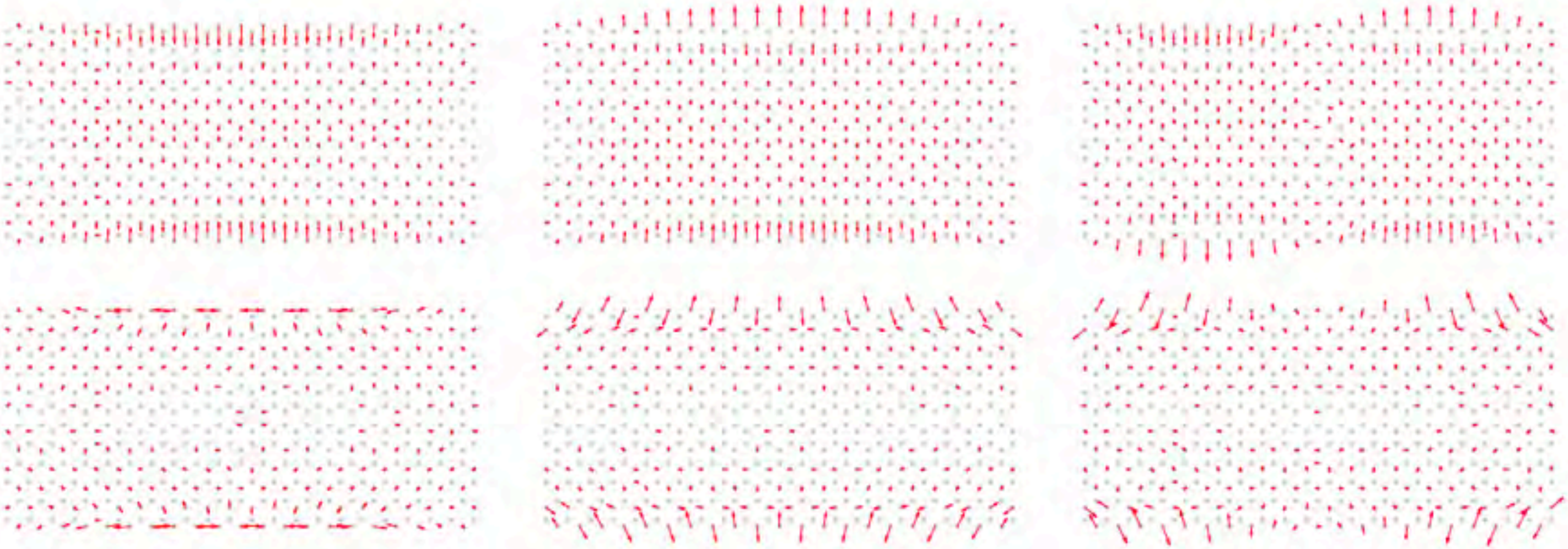
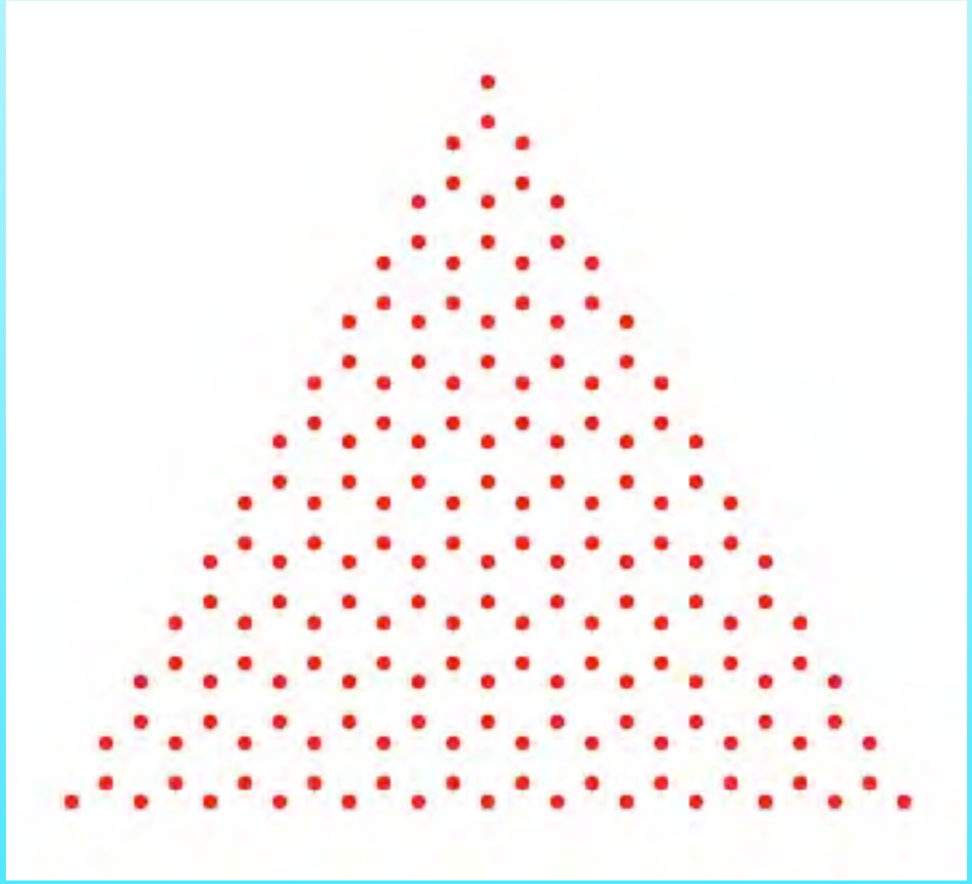


FIG. 5. Real space picture of the typical eigenmodes that are localized to the edge. Upper panels: $\eta = 1$. Lower panels: $\eta = 1/3$. For $\eta = 1$ ($\eta = 1/3$), the neighboring mass points at the edge oscillate in-phase (anti-phase).

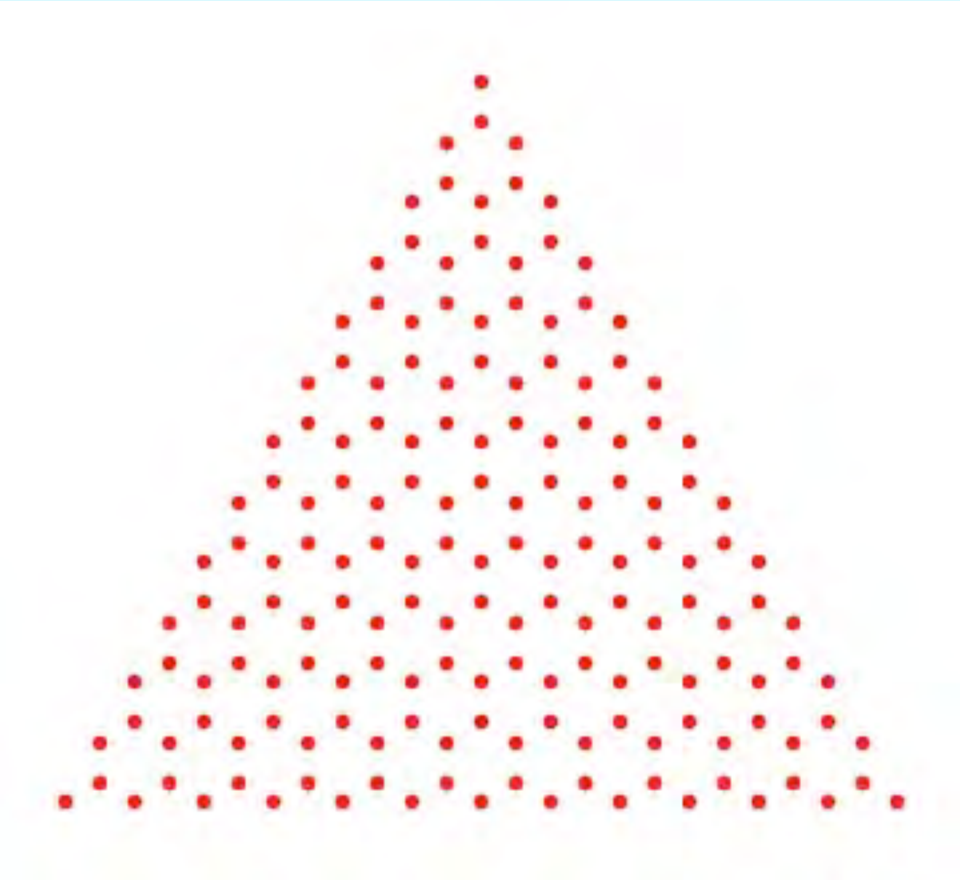
Edge waves with various shapes

With rotation: time-reversal symmetry breaking Coriolis force (effective magnetic field)

Topological change associated with Chern number jumps



$\eta = 1/3 \ (C = -2)$

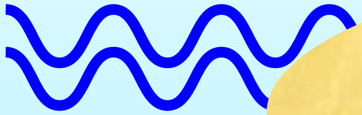


$\eta = 1 \ (C = 1)$

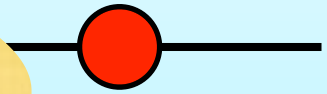
Bulk-Edge correspondence

Universality

YH, '93



Even in classical world !



Bulk state
(scattering state)
Bulk Gap
Non trivial Vacuum

with
each other

Edge state
(Bound state)
Particles in the gap

Edge states are
topological order parameters
accessible by experiments !

Use of the edge states

Edge states

localized particles in the gap

Novel quantum degrees
with topological protection by bulk

Conclusion

Thank you

Edge states are everywhere
in
condensed matter physics

Edge states are useful for
applications
in quantum physics /devices

Summary

2003 Talk at MIT

Topological Order

is NOT merely

FANCY

but

USEFUL!

Summary

2011 at Nagano, 2012 Hiroshima

Topology is now everywhere
in
condensed matter physics

2010 at Orland

Edge states are everywhere
in
condensed matter physics

Summary

2015 at Chiba

Topology

Symmetry

Thank you !

Edge states

Another physical way to look at matter !