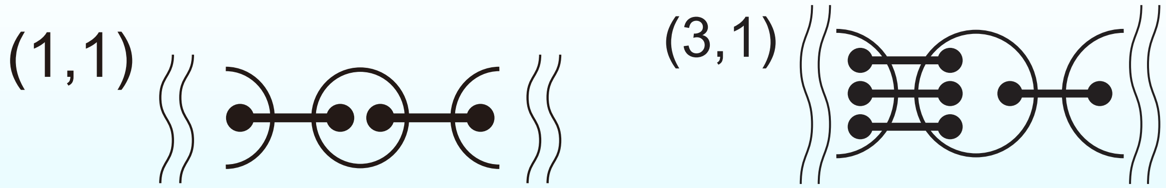
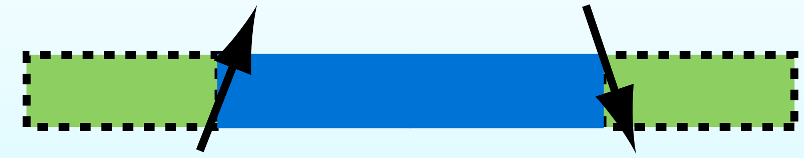
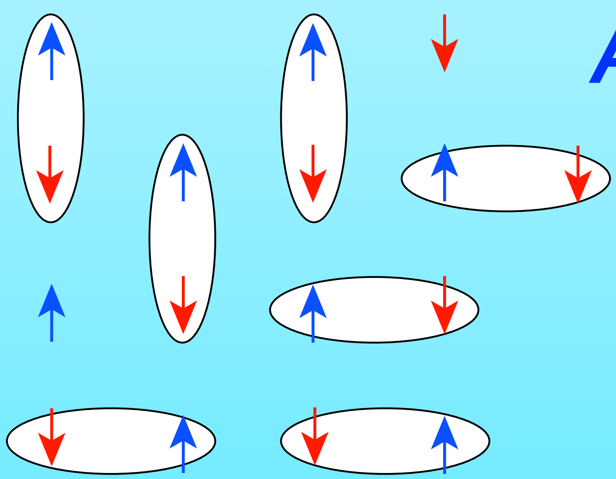


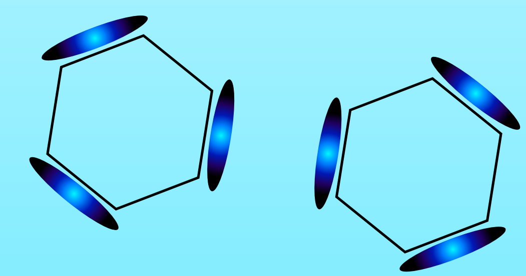
Dec. 10, 2008 at KITP



Bulk-Edge correspondence and Fractionalization

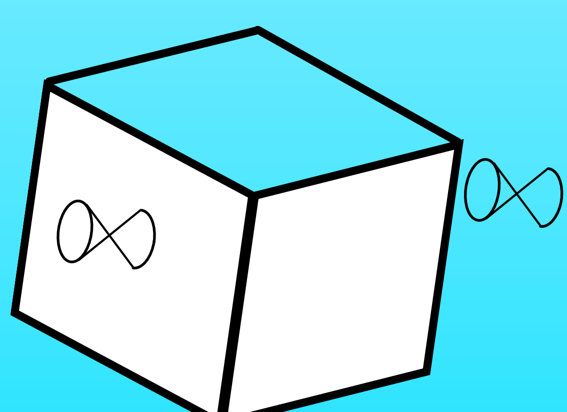
As a topological (spin) insulator with strong interaction





Y. Hatsugai
Institute of Physics
Univ. of Tsukuba
JAPAN

$$i\gamma_C(A_\psi) = \int_C A_\psi$$



University of Tsukuba
筑波大学

Plan

With time reversal invariance

- $\mathbb{Z}_2$ Berry phase for a topological order parameter
  - Fractionalization for the Bulk in 1D & 2D
- Entanglement Entropy to detect edge states
  - (effective) Description by the Edges :
    - Fractionalization at the Edges in 1D

deconfined spinons in 2D & 3D ??

- Time Reversal operators with interaction
  - Global to Local : super-selection rule $\Theta^2 = 1,$ or -1

Let us consider

Gapped spin liquid as a topological insulator

with strong interaction

Quantum Liquids without Symmetry Breaking

- Quantum Liquids in Low Dimensional Quantum Systems
  - Low Dimensionality, Quantum Fluctuations
  - No Symmetry Breaking
  - No Local Order Parameter
  - Various Phases & Quantum Phase Transitions

Topological Order

X.G.Wen

- Gapped Quantum Liquids in Condensed Matter
  - Integer & Fractional Quantum Hall States
  - Dimer Models of Fermions and Spins
  - Integer spin chains
  - Valence bond solid (VBS) states
  - Half filled Kondo Lattice

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How to understand gapped quantum liquids ?

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low energy localized modes in the gap

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edge states for QHE — Laughlin, Halperin, YH

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Common property of topological ordered states

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Edge

low energy localized modes in the gap

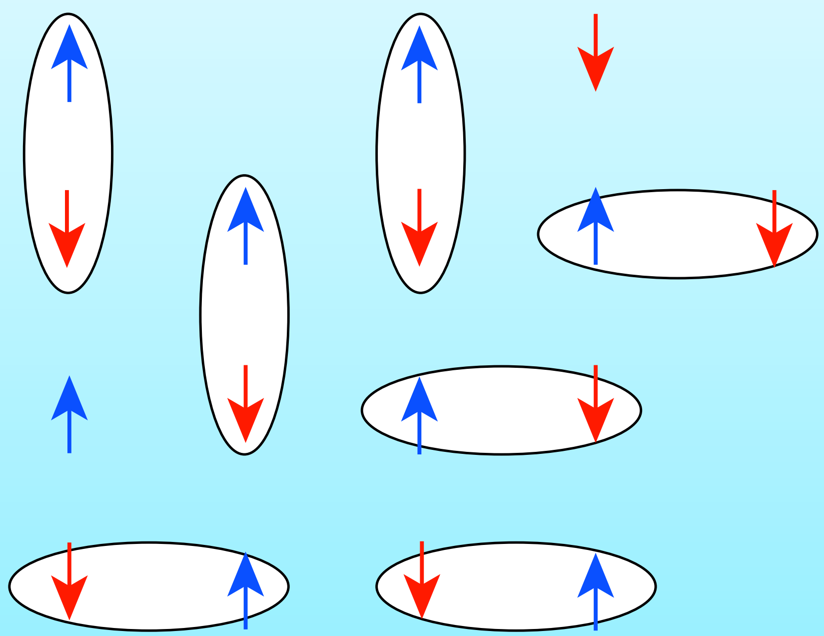
edge states for QHE Laughlin, Halperin, YH

As for quantum spins

- Z_2 Berry Phase as a Topological Order Parameter of bulk
- Entanglement Entropy to detect edge states (generic Kennedy triplet)

Quantum Liquid (Example 1)

☆ The *RVB* state by Anderson



$$|\text{Singlet Pair}_{12}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2$$

$$|G\rangle = \sum_{J=\text{Dimer Covering}} c_J \otimes_{ij} |\text{Singlet Pair}_{ij}\rangle$$

Local Singlet Pairs :

(Basic Objects)

Purely Quantum Objects are fundamental

Quantum Liquid (Example 1)

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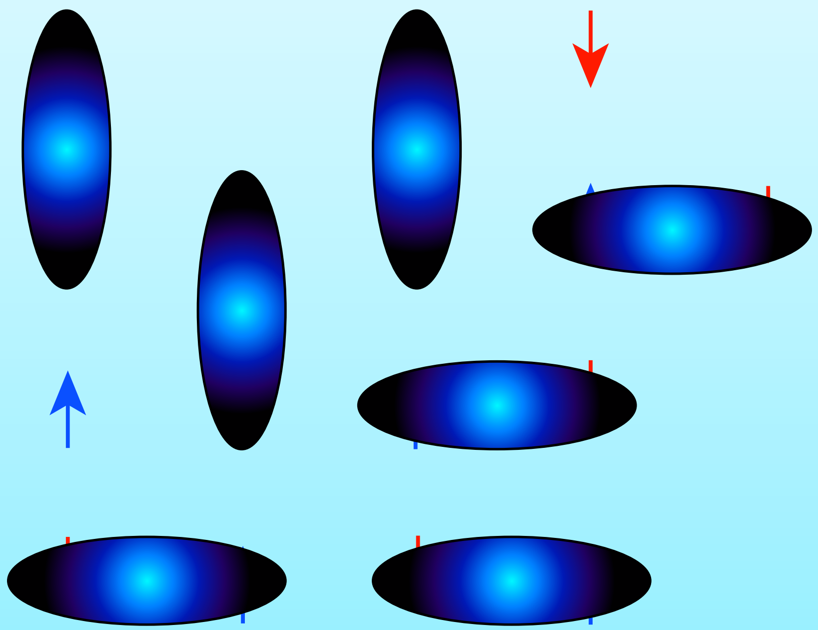

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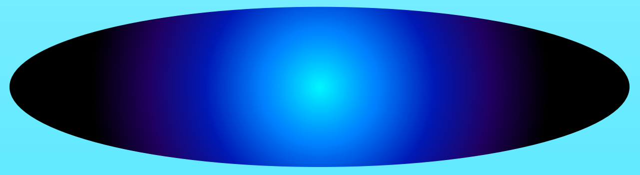
Spins disappear
as a Singlet pair



Local Singlet Pairs :
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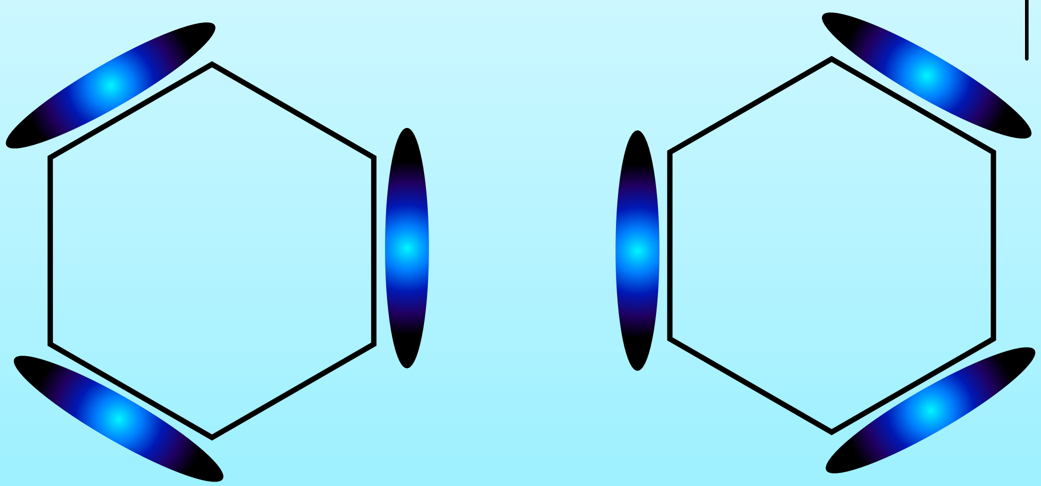
Quantum Liquid (Example 2)

☆The **RVB** state by Pauling



$$|\mathrm{Bond}_{12}\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) = \frac{1}{\sqrt{2}}(c_1^\dagger + c_2^\dagger)|0\rangle$$

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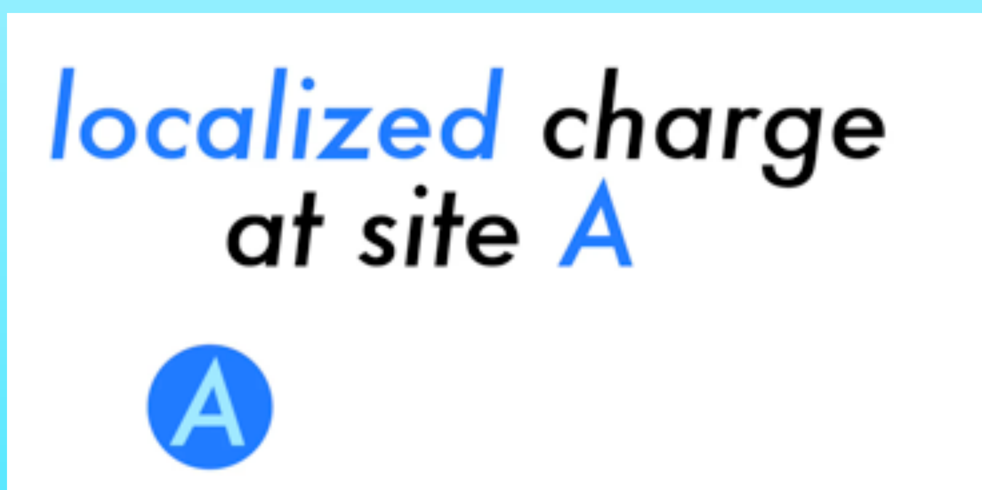
Do Not use the Fermi Sea



Local Covalent Bonds :

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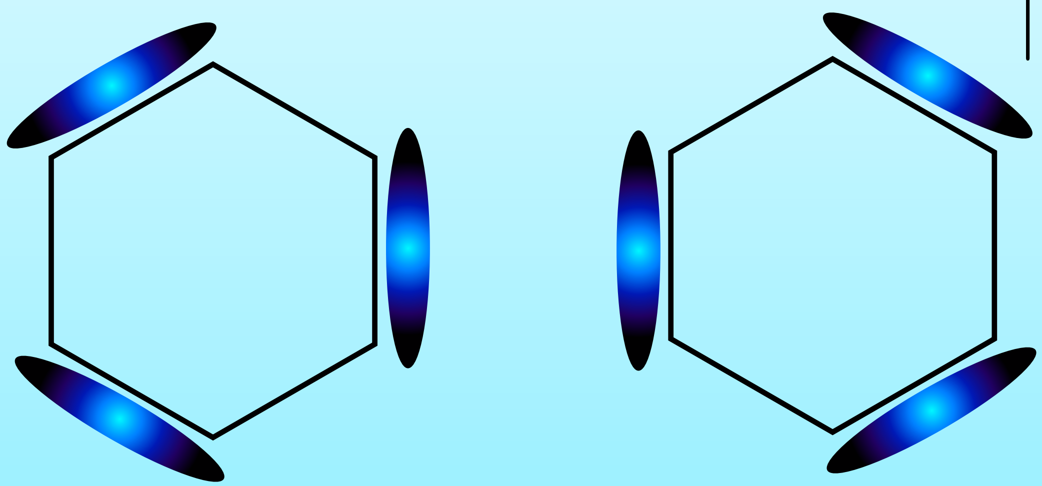
Quantum Liquid (Example 2)

☆ The RVB state by Pauling



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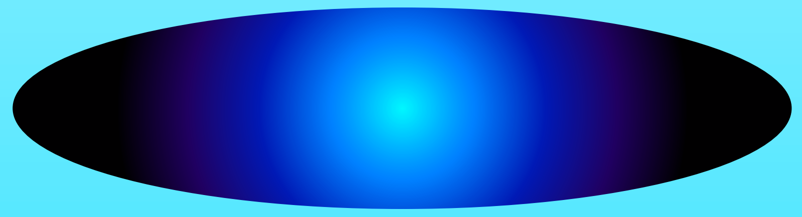
Do Not use the Fermi Sea

Delocalized charge
as a covalent bond



Local Covalent Bonds :
(Basic Objects)

Purely Quantum Objects are fundamental

Quantum Interference for the Classification

- "Classical" Observables
  - Charge density, Spin density,...

$$\mathcal{O} = n_\uparrow \pm n_\downarrow, \cdots$$

$$\langle \mathcal{O} \rangle_G = \langle G | \mathcal{O} | G \rangle = \langle G' | \mathcal{O} | G' \rangle = \langle \mathcal{O} \rangle_{G'}$$

$$|G'\rangle = |G\rangle e^{i\phi}$$

- "Quantum" Observables !
  - Quantum Interferences:

$$\langle G_1 | G_2 \rangle = \langle G_1' | G_2' \rangle e^{i(\phi_1 - \phi_2)}$$

  - Probability Amplitute (overlap)

$$|G_i\rangle = |G_i'\rangle e^{i\phi_i}$$

    - Aharonov-Bohm Effects
    - Phase (Gauge) dependent

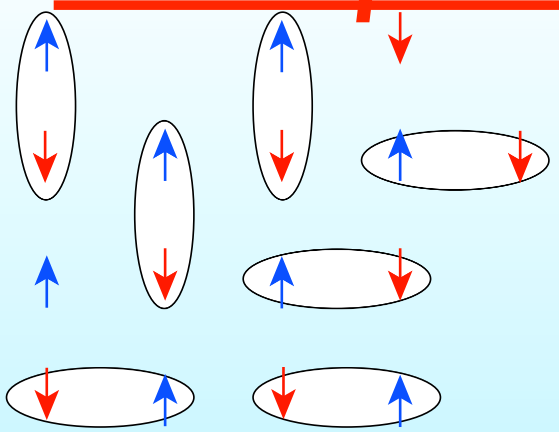
$$\langle G | G + dG \rangle = 1 + \langle G | dG \rangle$$

$A = \langle G | dG \rangle$:Berry Connection

$i\gamma = \int A$:Berry Phase

Use Quantum Interferences To Classify Quantum Liquids

Examples: RVB state by Anderson



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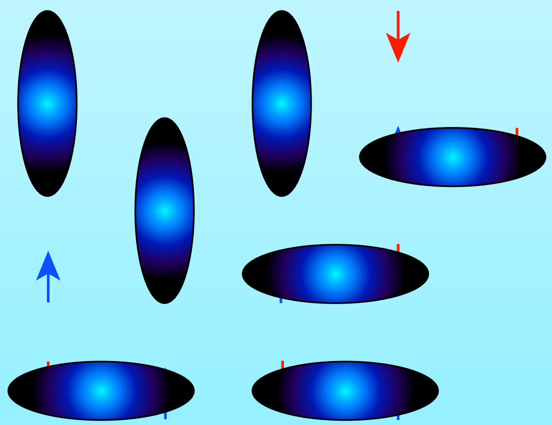
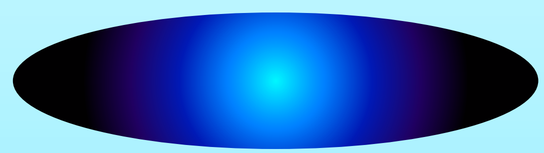


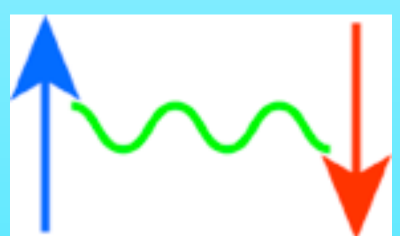
Spins *disappear*
as a *Singlet pair*



No Long Range Order
in Spin-Spin Correlation

Local Singlet Pair is a Basic Object

How to Characterize the Local Singlet Pair ?



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Use Berry Phase to characterize the Singlet!

Singlet does not carries spin but does Berry phase

$$\gamma_{\text{singlet pair}} = \pi \mod 2\pi$$

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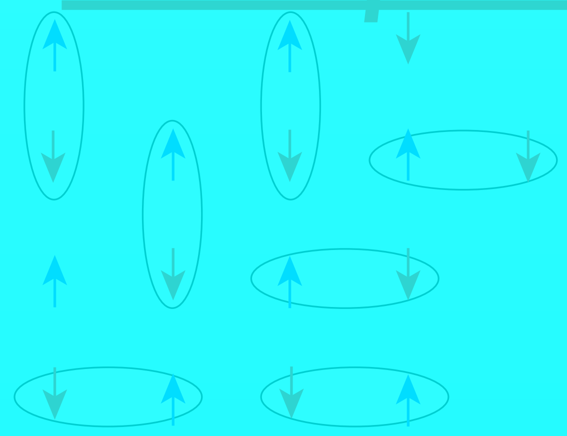

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Spins disappear as a Singlet pair

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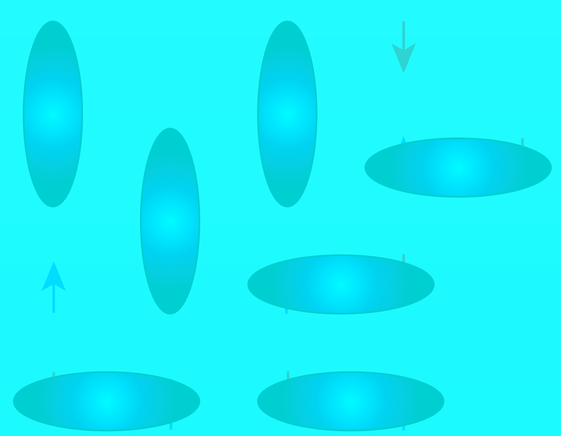

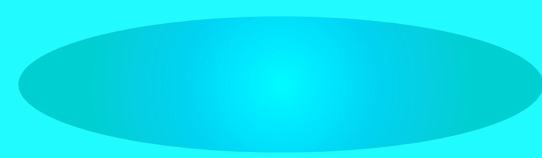

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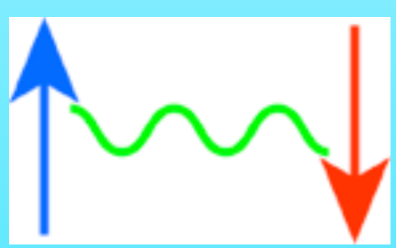

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Z_2 Berry phases for gapped quantum spins

- generic Heisenberg Models (with frustration)

$$H = \sum_{ij} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j$$

Time Reversal Invariant

$$\Theta_N \boldsymbol{S}_i \Theta_N^{-1} = -\boldsymbol{S}_i$$

$$[H, \Theta_N] = 0$$

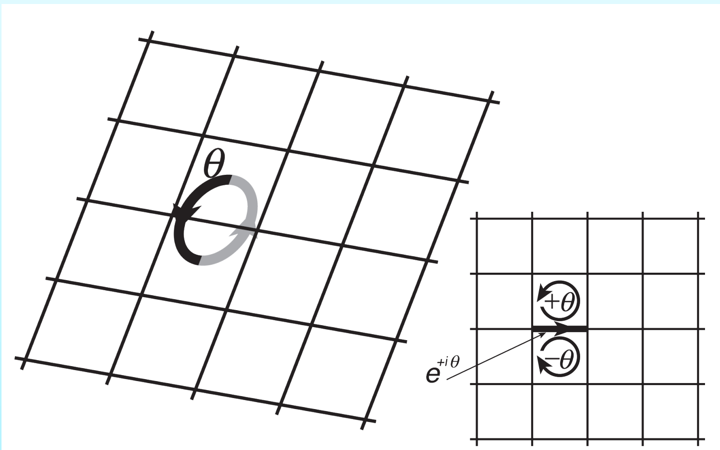
$$\Theta_N = (i\sigma_y^1) \otimes (i\sigma_y^2) \cdots (i\sigma_y^N) K$$

$$\Theta_N^2 = (-)^N$$

Mostly N: even $\Theta_N^2 = 1$ (probability 1/2 in HgTe)

Z_2 Berry phases for gapped quantum spins

Define a many body hamiltonian by local twist as a parameter



$$H(x = e^{i\theta})$$

$$C = \{x = e^{i\theta} | \theta : 0 \to 2\pi\}$$ U(1)

$$\boldsymbol{S}_i \cdot \boldsymbol{S}_j \to \frac{1}{2}(e^{-i\theta} S_{i+} S_{j-} + e^{+i\theta} S_{i-} S_{j+}) + S_{iz} S_{jz}$$

Only link <ij>

Calculate the Berry Phases using the Entire Many Spin Wavefunction numerically

Require excitation Gap!

Z_2 quantization

$$\gamma_C = \int_C A_\psi = \int_C \langle \psi | d\psi \rangle = \begin{cases} 0 \\ \pi \end{cases} : \mod 2\pi \qquad Z_2$$

Time Reversal (Anti-Unitary) Invariance

Berry Connection and Gauge Transformation

- **Parameter Dependent Hamiltonian**

$H(x)\psi(x)=E(x)\psi(x)$

$$H(x)|\psi(x)\rangle = E(x)|\psi(x)\rangle,\ \langle\psi(x)|\psi(x)\rangle = 1.$$

- **Berry Connections** $A_\psi = \langle\psi|d\psi\rangle = \langle\psi|\frac{d}{dx}\psi\rangle dx.$
- **Berry Phases** $i\gamma_C(A_\psi) = \int_C A_\psi$

(Abelian)

- **Phase Ambiguity of the eigen state**

Gauge Transformation

$$|\psi(x)\rangle = |\psi'(x)\rangle e^{i\Omega(x)}$$

$$A_\psi = A'_\psi + id\Omega = A'_\psi + i\frac{d\Omega}{dx}dx$$

- **Berry phases are not well-defined without specifying the gauge**

$$\gamma_C(A_\psi) = \gamma_C(A_{\psi'}) + \int_C d\Omega$$

$2\pi\times(\text{integer})$ if $e^{i\Omega}$ is single valued

- **Well Defined up to mod 2π**

$$\gamma_C(A_\psi) \equiv \gamma_C(A_{\psi'}) \mod 2\pi$$

Anti-Unitary Operator and Berry Phases

☆ Anti-Unitary Operator (Time Reversal, Particle-Hole)

$$\Theta = K U_\Theta, \quad \begin{array}{rl} K: & \text{Complex conjugate} \\ U_\Theta: & \text{Unitary} \end{array}$$

(parameter independent)

$$|\Psi\rangle = \sum_J C_J |J\rangle \qquad \sum_J C_J^* C_J = \langle\Psi|\Psi\rangle = 1$$

$$|\Psi^\Theta\rangle = \Theta|\Psi\rangle = \sum_J C_J^* |J^\Theta\rangle, \quad |J^\Theta\rangle = \Theta|J\rangle$$

☆ Berry Phases and Anti-Unitary Operation

$$A^\Psi = \langle\Psi|d\Psi\rangle = \sum_J C_J^* dC_J \qquad \sum_J dC_J^* C_J + \sum_J C_J^* dC_J = 0$$

$$A^{\Theta\Psi} = \langle\Psi^\Theta|d\Psi^\Theta\rangle = \sum_J C_J dC_J^* = -A^\Psi$$

$$\gamma_C(A^{\Theta\Psi}) = -\gamma_C(A^\Psi)$$

Anti-Unitary Invariant State and Z_2 Berry Phase

$\Theta_N^2 = 1$

- Anti-Unitary Symmetry $[H(x), \Theta] = 0$
- Invariant State ${}^\exists\varphi, \quad |\Psi^\Theta\rangle = \Theta|\Psi\rangle = |\Psi\rangle e^{i\varphi}$
  - ex. Unique Eigen State $\simeq |\Psi\rangle$ Gauge Equivalent(Different Gauge)
- To be compatible with the ambiguity,

  the Berry Phases have to be quantized as

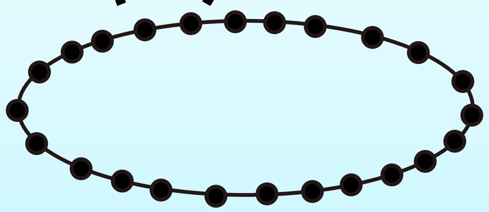
$$\gamma_C(A^\Psi) = \begin{cases} 0 \\ \pi \end{cases} \mod 2\pi$$

$\mathbb{Z}_2$ Berry phase

$$\gamma_C(A^\Psi) = -\gamma_C(A^{\Theta\Psi}) \equiv -\gamma_C(A^\Psi), \ \mathrm{mod} 2\pi$$

Numerical Evaluation of the Berry Phases (incl. non-Abelian)

(1) Discretize the periodic parameter space



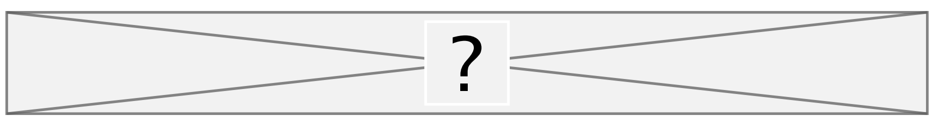
$x_0, x_1, \cdots, x_N = x_0$ $\quad$ $\theta_0 = 0,\ \theta_N = 2\pi$

$x_n = e^{i\theta_n}$ $\quad$ $\theta_{n+1} = \theta_n + \Delta\theta_n$ $\quad$ ${}^\forall \Delta\theta_n \to 0$

(2) Obtain eigen vectors

$$H(x_n)|\psi_n^i\rangle = E^i(x_n)|\psi_n^i\rangle$$

(3) Define Berry connection in a discretized form



non-Abelian $A_n = \mathrm{Im}\log\det D_n,\ \{D_n\}_{ij} = \langle\psi_n^i|\psi_{n+1}^j\rangle$

(4) Evaluate the Berry phase

non-Abelian

$$\gamma = \sum_{n=0}^{N-1} A_n = \mathrm{Im}\log\langle\psi_0|\psi_1\rangle\langle\psi_1|\psi\rangle\cdots (= \mathrm{Im}\log\det D_1 D_2 \cdots D_n)$$

Independent of the choice of the phase $\quad |\psi_n\rangle \to |\psi_n\rangle' e^{i\Omega_n}$

Gauge invariant

after the discretization

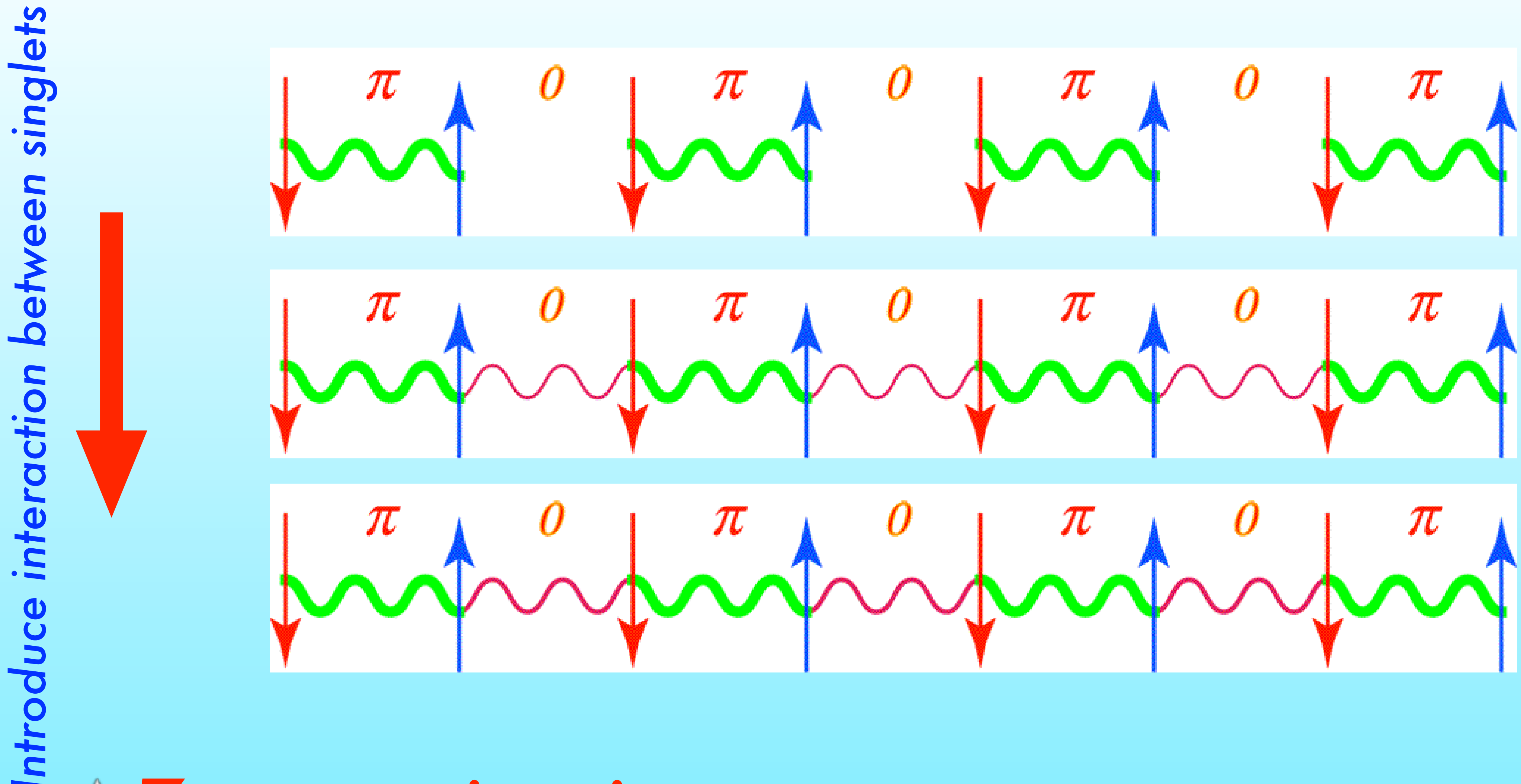
Convenient for Numerics

Luscher '82 (Lattice Gauge Theory)

King-Smith & Vanderbilt '93 (polarization in solids)

T. Fukui, H. Suzuki & YH '05 (Chern numbers)

Adiabatic Continuation & the Quantization



☆ Z_2-quantization of the Berry phases protects from continuous change

Adiabatic Continuation in a gapped system

↕

Renormalization Group in a gapless system

Local Order Parameters of Singlet Pairs

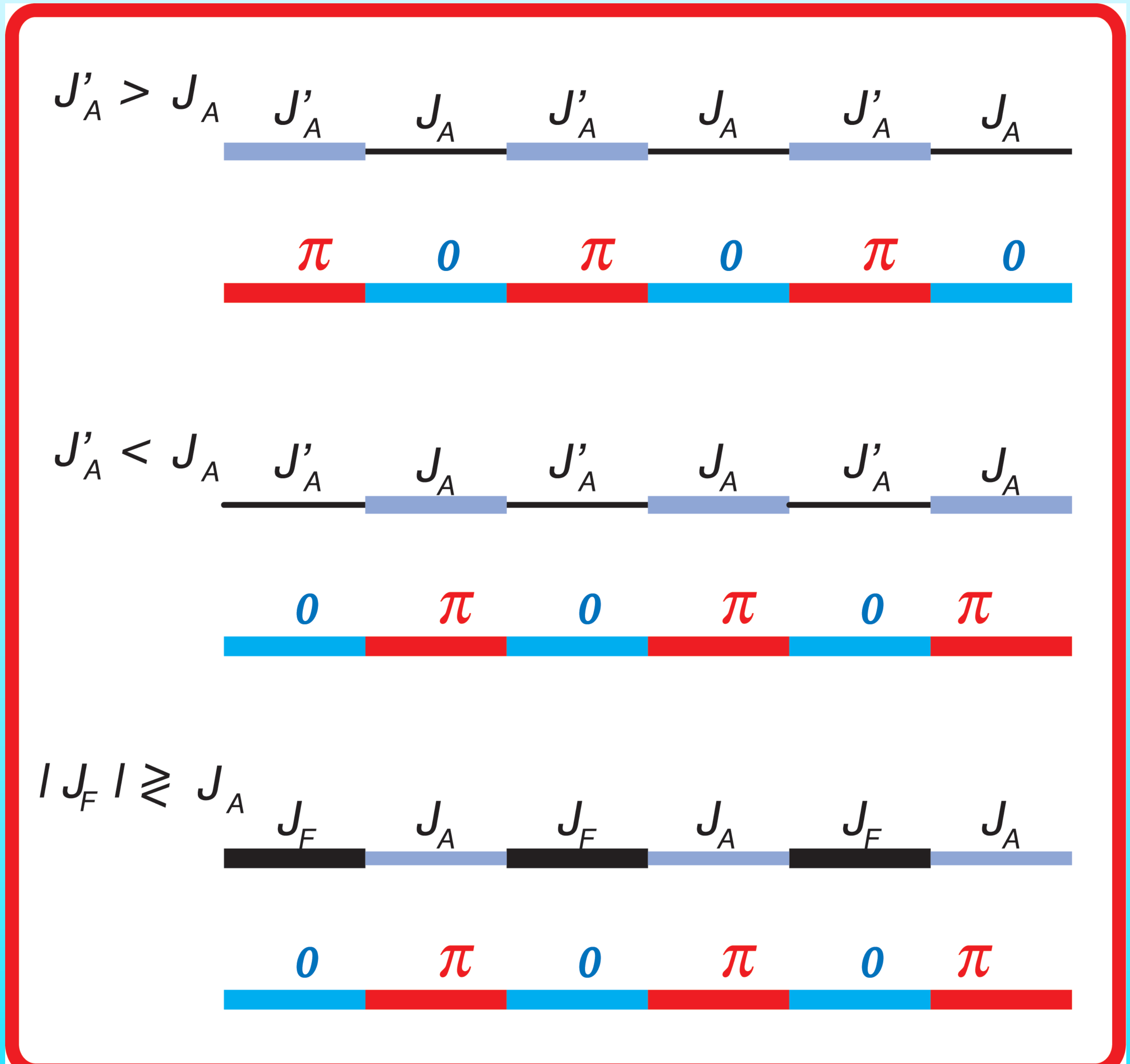
☆ 1D AF-AF, AF-F Dimers — Y.H., J. Phys. Soc. Jpn. 75 123601 (2006)

- ☆ *Strong Coupling Limit of the AF Dimer link is a gapped unique ground state.*

AF-AF

F-AF

Hida



AF-AF case

Strong bonds

: π bonds

F-AF case

AF bonds

: π bonds

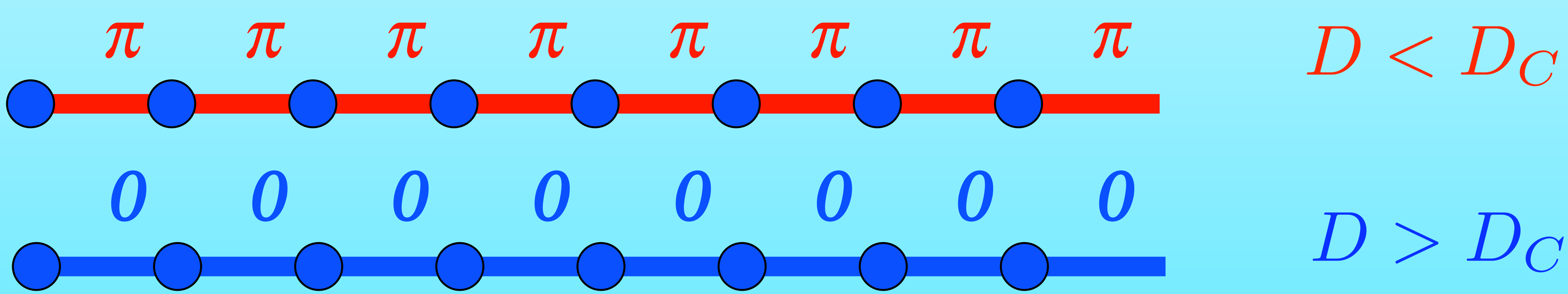
Local Order Parameters of the Haldane Phase

- Heisenberg Spin Chains with integer S
  - No Symmetry Breaking by the Local Order Parameter
  - "String Order": Non-Local Order Parameter!

S=1 $(\boldsymbol{S}_i)^2 = S(S+1),\ S=1$

$$H = J\sum_{\langle ij\rangle} \boldsymbol{S}_i\cdot\boldsymbol{S}_j + D\sum_i (S_i^z)^2$$

Y.H., J. Phys. Soc. Jpn. 75 123601 (2006)



Describe the Quantum Phase Transition locally

c.f. S=1/2, 1D dimers, 2D with Frustrations, Ladders
t-J with Spin gap

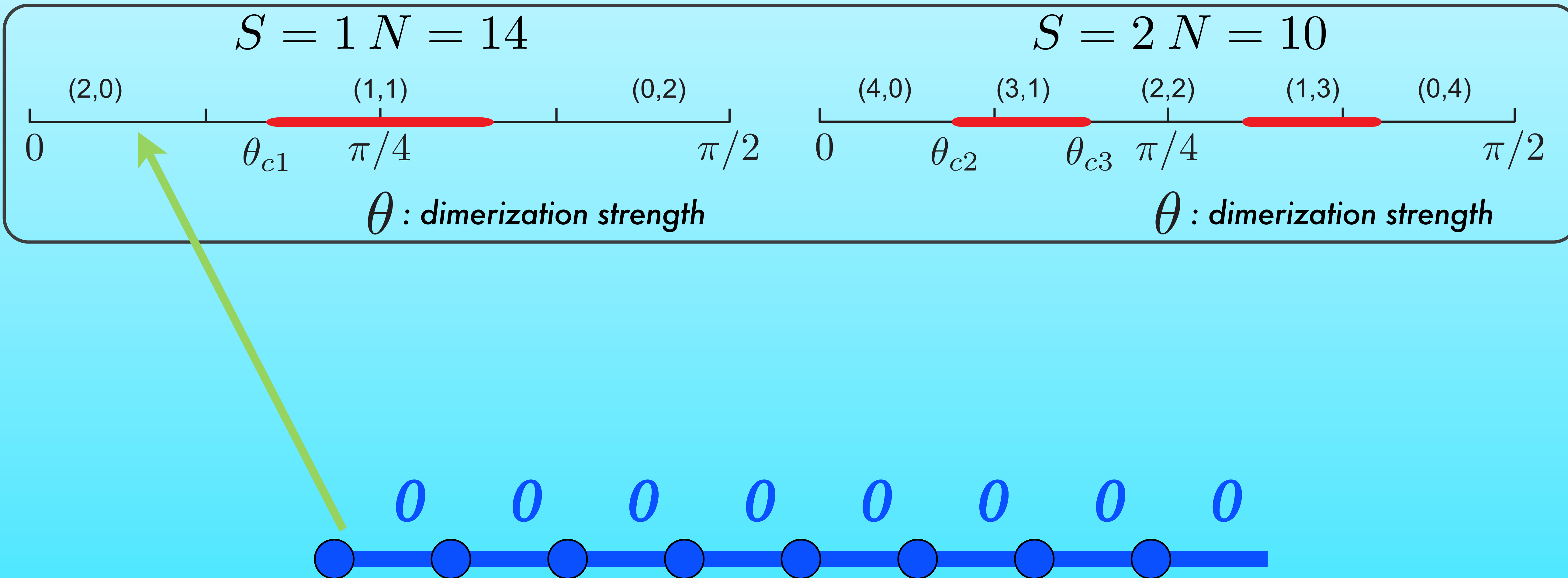
Topological Classification of Gapped Spin Chains

T.Hirano, H.Katsura &YH, Phys.Rev.B77 094431'08

- S=1,2 dimerized Heisenberg model

$$H = \sum_{i=1}^{N/2} (J_1 \boldsymbol{S}_{2i} \cdot \boldsymbol{S}_{2i+1} + J_2 \boldsymbol{S}_{2i+1} \cdot \boldsymbol{S}_{2i+2}) \qquad J_1 = \cos\theta, J_2 = \sin\theta$$

Z₂Berry phase



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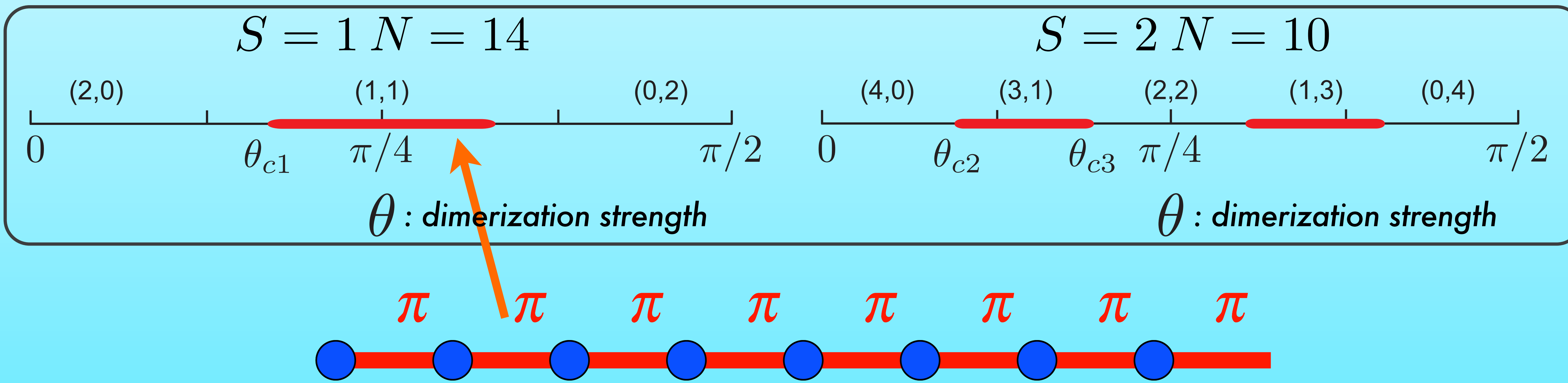
Z_2 Berry phase



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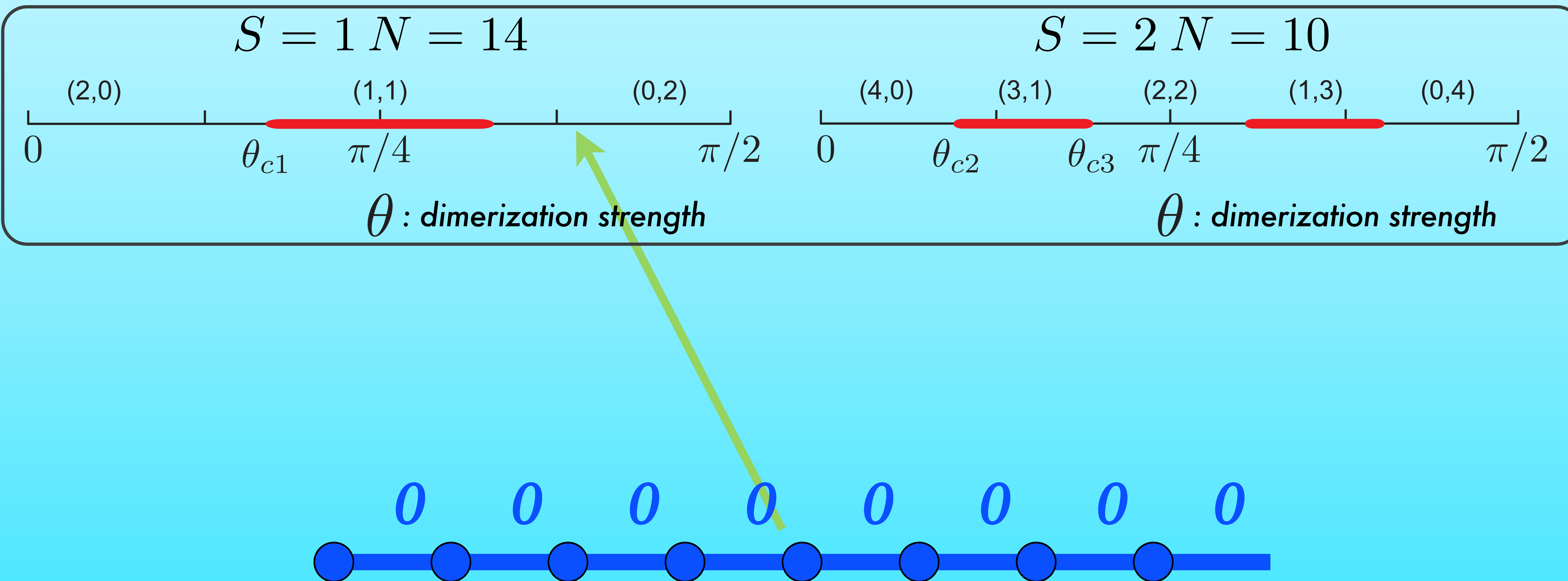

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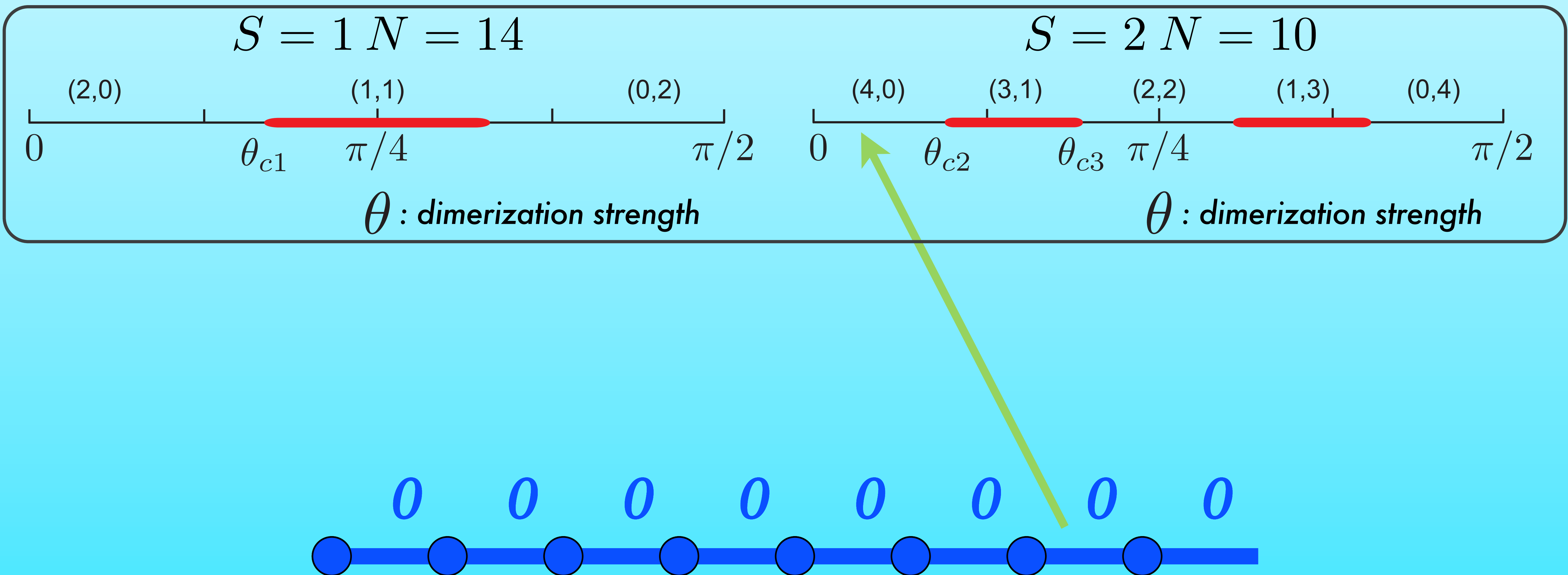

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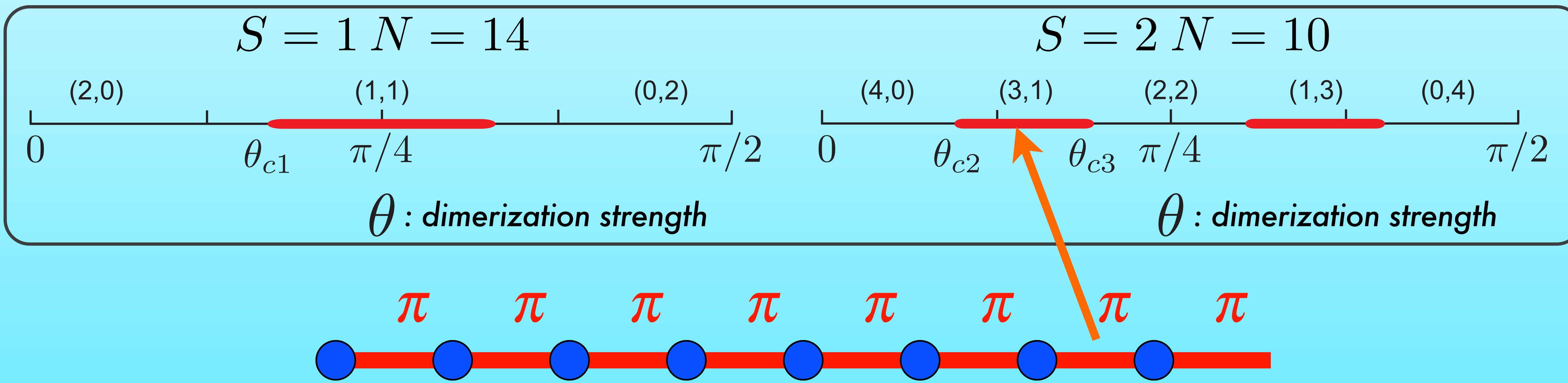
$$H = \sum_{i=1}^{N/2} (J_1 \boldsymbol{S}_{2i} \cdot \boldsymbol{S}_{2i+1} + J_2 \boldsymbol{S}_{2i+1} \cdot \boldsymbol{S}_{2i+2}) \qquad J_1 = \cos\theta, J_2 = \sin\theta$$

Z_2 Berry phase



Topological Classification of Gapped Spin Chains

T.Hirano, H.Katsura &YH, Phys.Rev.B77 094431'08

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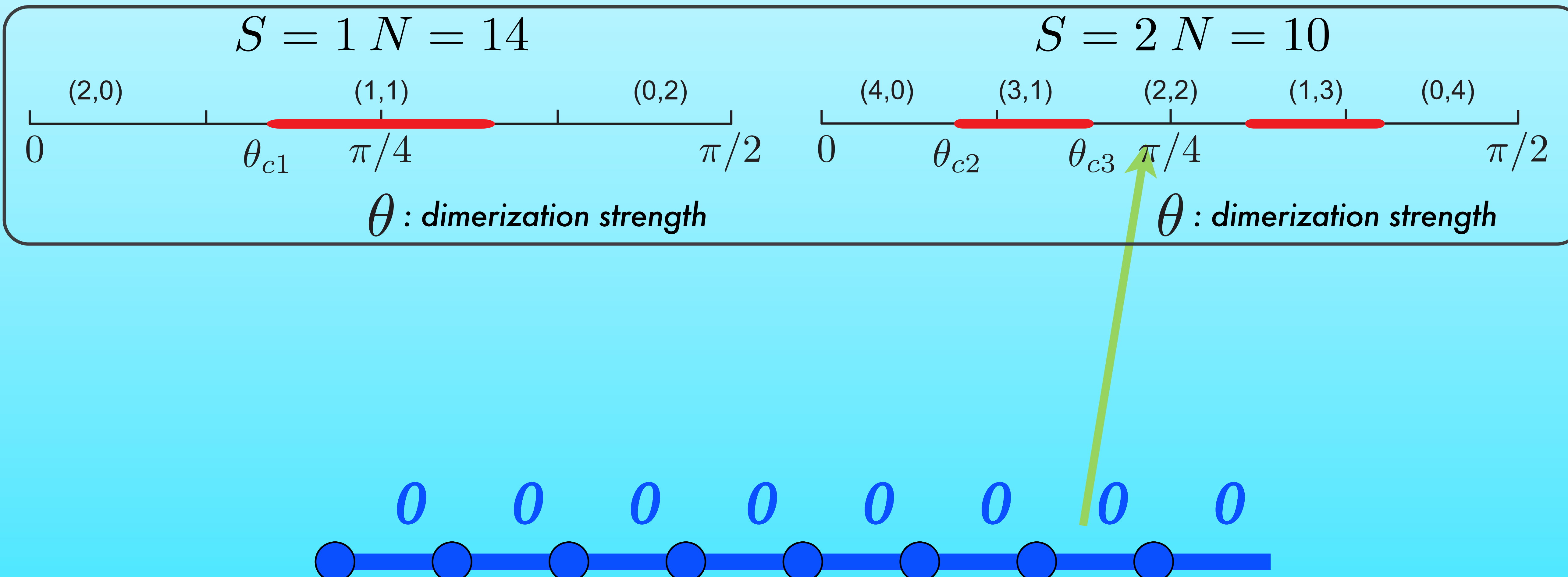

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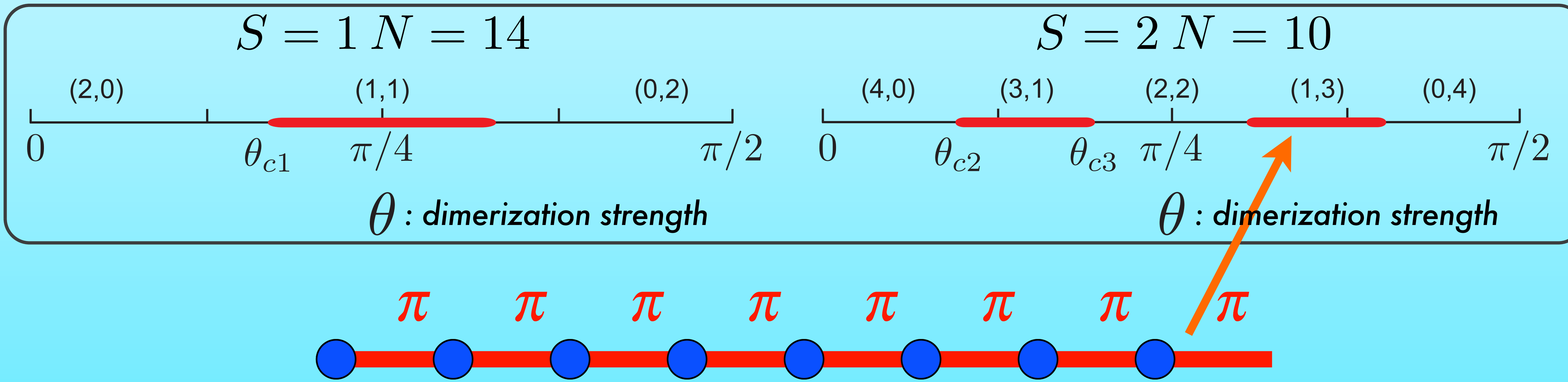
Z₂Berry phase



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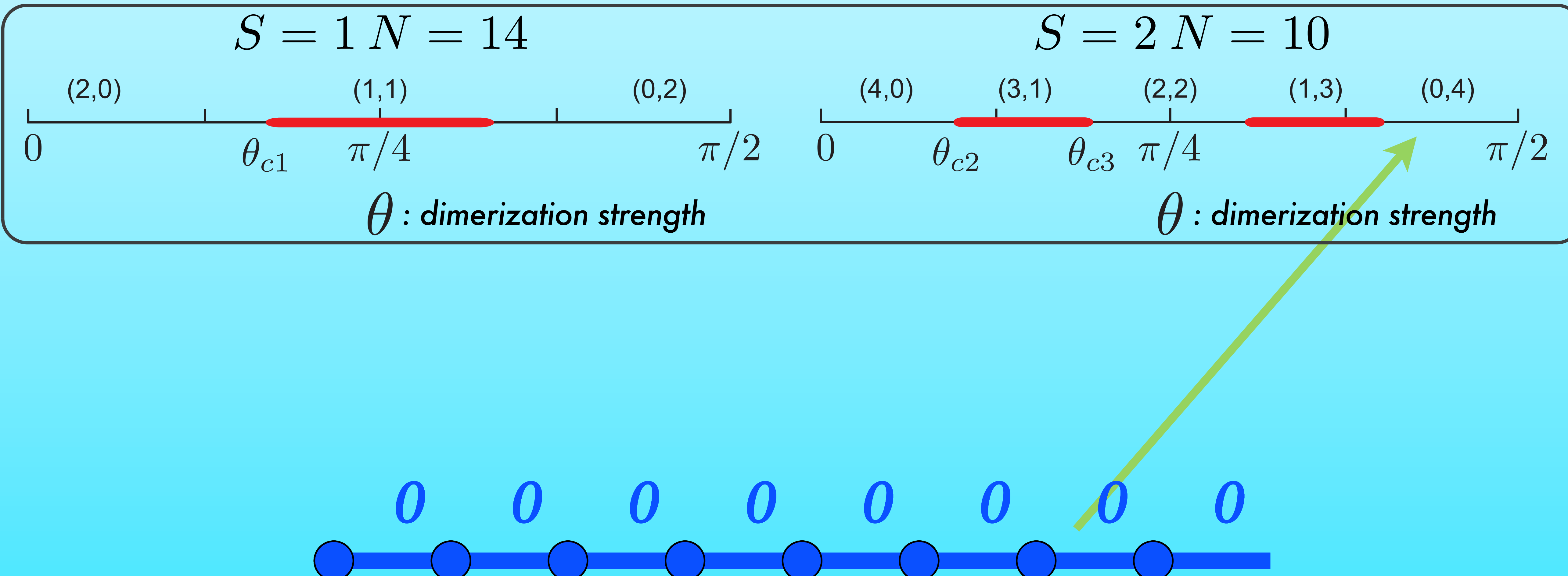

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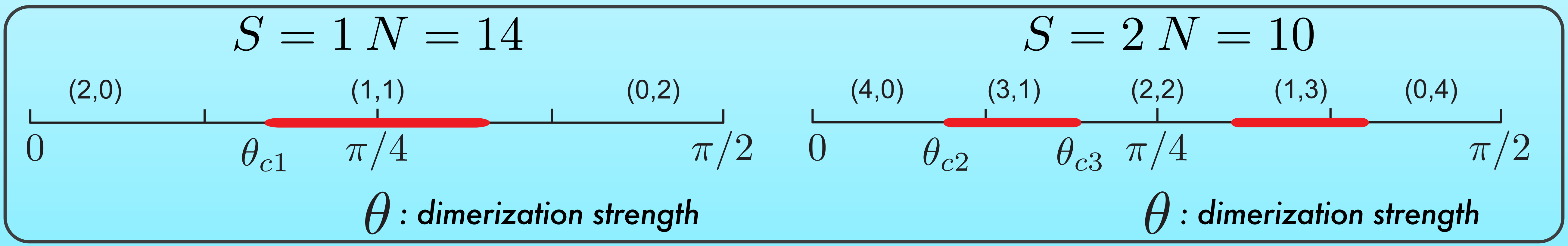
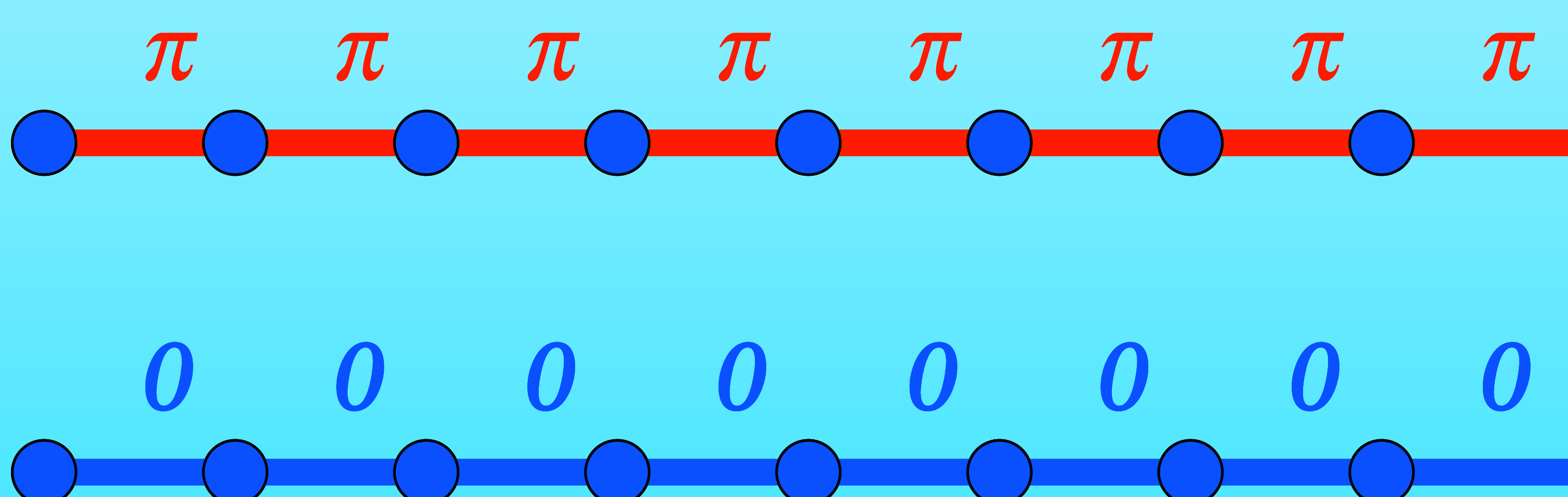




Topological Quantum Phase Transitions with translation invariance

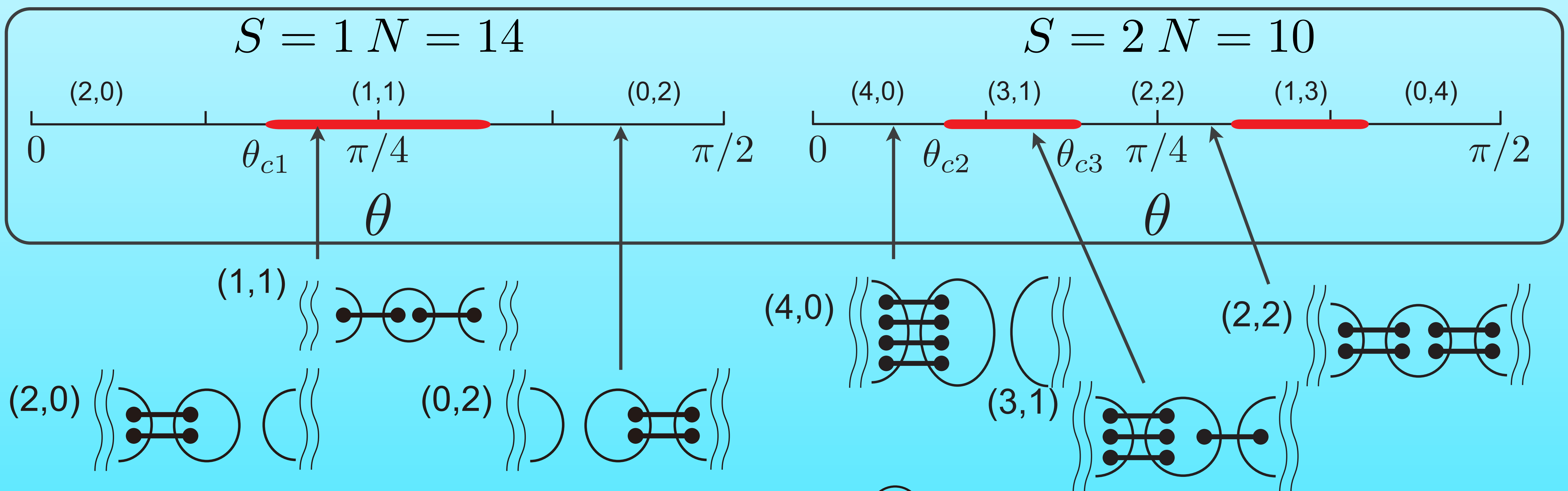
Topological Classification of Gapped Spin Chains

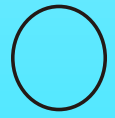
T.Hirano, H.Katsura &YH, Phys.Rev.B77 094431'08

- S=1,2 dimerized Heisenberg model

$$H = \sum_{i=1}^{N/2} (J_1 \boldsymbol{S}_{2i} \cdot \boldsymbol{S}_{2i+1} + J_2 \boldsymbol{S}_{2i+1} \cdot \boldsymbol{S}_{2i+2}) \qquad J_1 = \cos\theta, J_2 = \sin\theta$$

Z_2 Berry phase



 : S=1/2 singlet state  : Symmetrization

Reconstruction of valence bonds!

Topological Classification of Gapped Spin Chains (cont.)

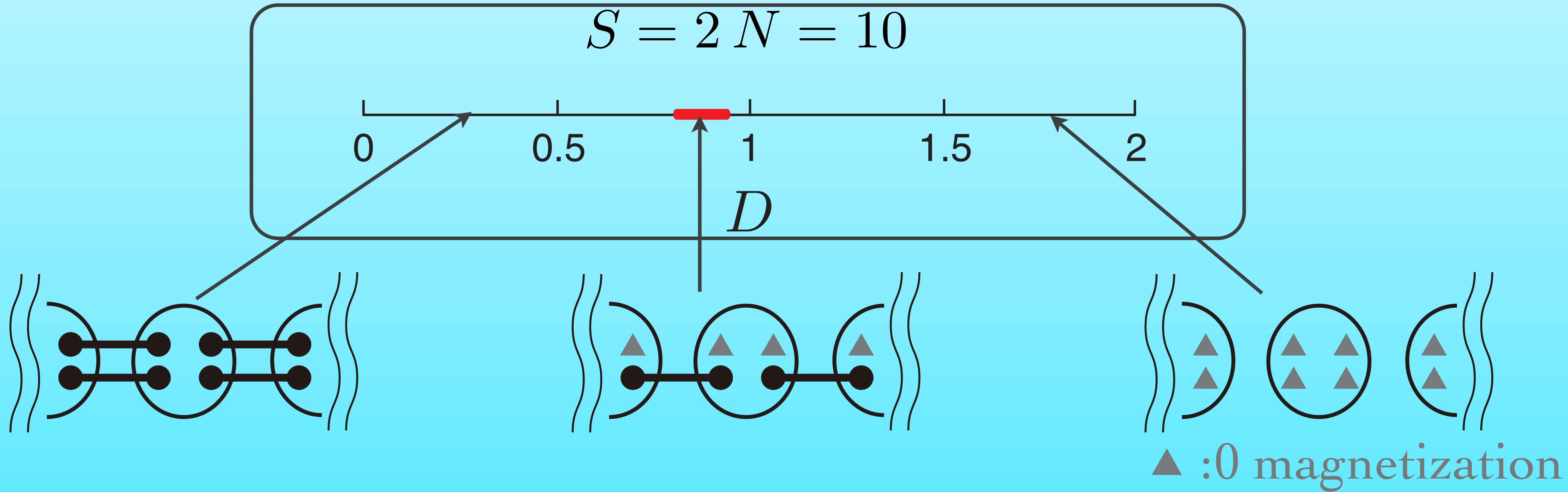
T.Hirano, H.Katsura &YH, Phys.Rev.B77 094431'08

- S=2 Heisenberg model with D-term

$$H = \sum_{i}^{N} \left[J \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + D \left(S_i^z \right)^2 \right]$$

Z_2 Berry phase

Red line denotes the non trivial Berry phase



Reconstruction of valence bonds!

Topological Classification of Generic AKLT (VBS) models

T.Hirano, H.Katsura &YH, Phys.Rev.B77 094431'08

Twist the link of the generic AKLT model

$$H(\{\phi_{i,i+1}\}) = \sum_{i=1}^{N} \sum_{J=B_{i,i+1}+1}^{2B_{i,i+1}} A_J P^J_{i,i+1}[\phi_{i,i+1}]$$

$$|\{\phi_{i,j}\}\rangle = \prod_{\langle ij \rangle} \left(e^{i\phi_{ij}/2} a_i^\dagger b_j^\dagger - e^{-i\phi_{ij}/2} b_i^\dagger a_j^\dagger \right)^{B_{ij}} |\text{vac}\rangle$$

Berry phase on a link (ij)

$$\gamma_{ij} = B_{ij}\pi \mod 2\pi$$

S=1/2

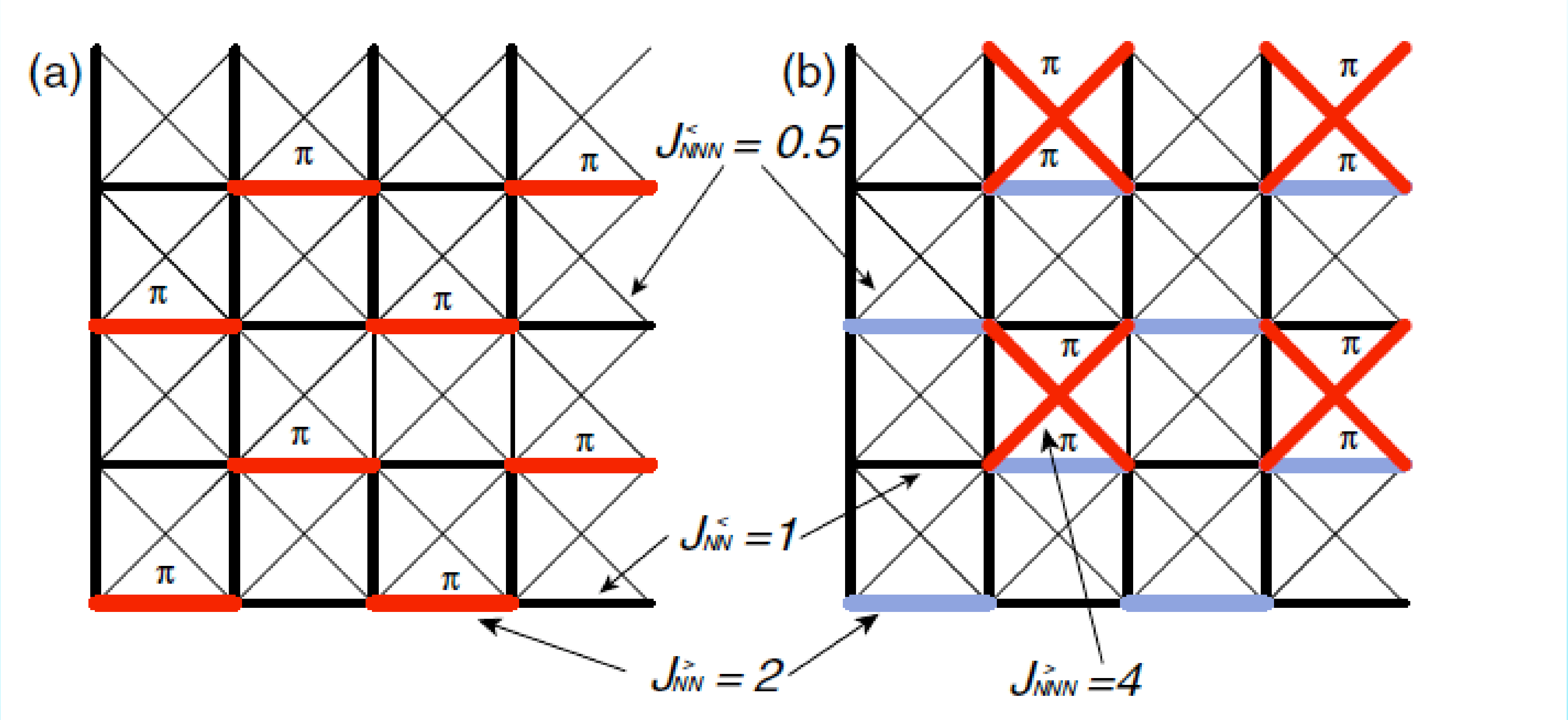
The Berry phase counts the number of the valence bonds!

S=1/2 objects are fundamental in S=1&2 spin chains

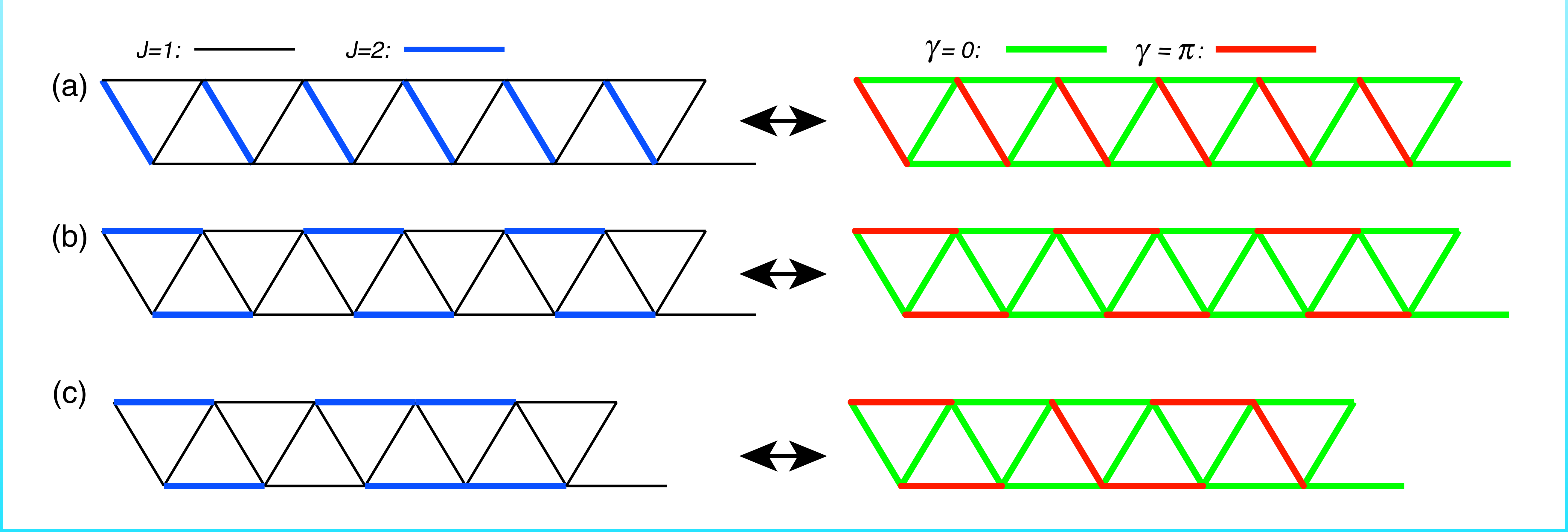
→ FRACTIONALIZATION

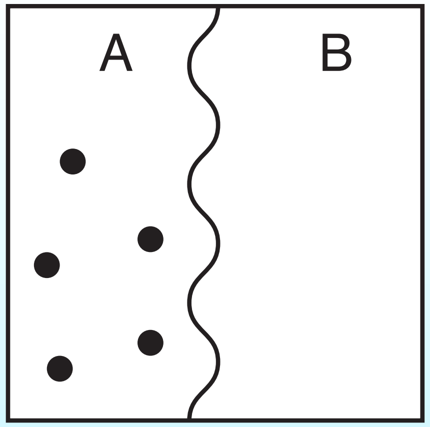
Contribute to the Entanglement Entropy as of Edge states

2D, Ladders (S=1/2), t-J (spin gapped)



Y.H., J. Phys. Soc. Jpn. 75 123601 (2006), J. Phys. Cond. Matt.19, 145209 (2007)





- ☑ *Entanglement Entropy to detect edge states*
- ☑ *direct calculation of spectrum with bondaries*

Entanglement Entropy

☆ Mixed State From Entanglement

Vidal, Latorre, Rico, Kitaev '02

☆ Direct Product State

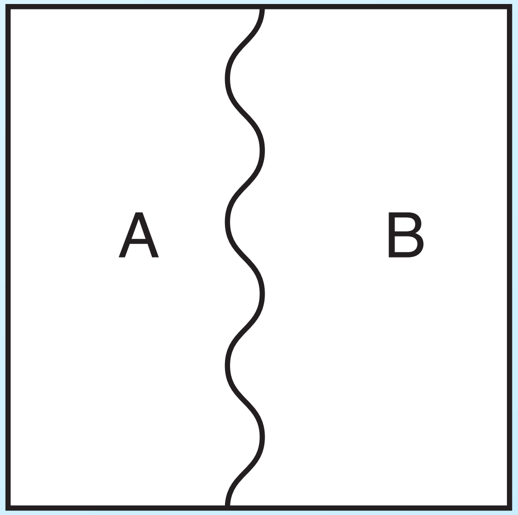
$$|\Psi_{AB}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$$

☆ Entangled State

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{D}} \sum_j^D |\Psi_A^j\rangle \otimes |\Psi_B^j\rangle$$

$$\text{System} = A \oplus B$$

$$\text{State} = \sum \Psi_A \otimes \Psi_B$$



☆ Partial Trace

$$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$$

$$\rho_A = \mathrm{Tr}_B\, \rho_{AB}$$

$$= \frac{1}{D} \sum_j^D |\Psi_A^j\rangle\langle\Psi_A^j|$$

Pure State

$D = 1$

Mixed State

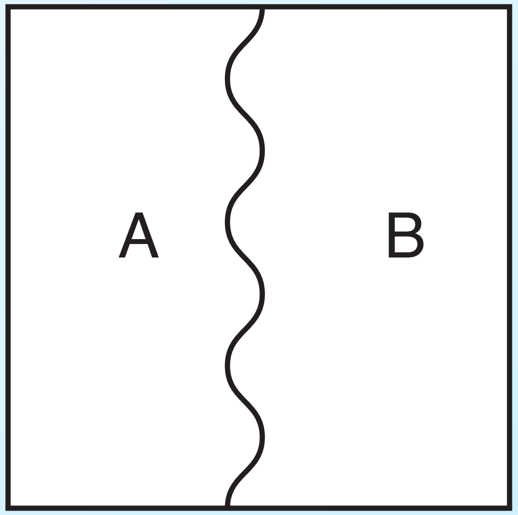
$$\rho_{AB} = \frac{1}{D} \sum_{jk}^D |\Psi_A^j\rangle\langle\Psi_A^k| \otimes |\Psi_B^j\rangle\langle\Psi_B^k|$$

☆ How much the State is Entangled between A & B?

Entanglement Entropy :

$$S_A = -\langle \log \rho_A \rangle = \log D$$

E.E. & Edge states (Gapped)
(of spins, fermions...)



- *Partial Trace induces effective edge states*
  - Requirement: Finite Energy Gap for the Bulk
- *The effective edge states contribute to the E.E.*
  - Let us assume that the edge states has degrees of freedom D_E

Entanglement Entropy > (# edge states) Log D_E

S. Ryu & YH, Phys. Rev. B73, 245115 (2006)
(Fermions)

EE of the Generic VBS States (S=1,2,3,...)

H. Katsura, T.Hirano & YH, Phys. Rev. B76, 012401 (2007)
T.Hirano & YH, J. Phys. Soc. Jpn. 76, 113601 (2007)

$$H_{VBS} = \sum_{i=1}^{N} \vec{S}_i \cdot \vec{S}_{i+1} + \alpha H^{S}_{\rm extra}, \qquad \vec{S}_i^2 = S(S+1)$$

$$H^{S=1}_{\rm extra} = \sum_i \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2$$

$$H^{S=2}_{\rm extra} = \sum_i \left(\frac{2}{9}(\vec{S}_i \cdot \vec{S}_{i+1})^2 + \frac{1}{63}(\vec{S}_i \cdot \vec{S}_{i+1})^3 \right.$$

$$|{\rm VBS}\rangle = \prod_{j=0}^{L} (a_j^\dagger b_{j+1}^\dagger - b_j^\dagger a_{j+1}^\dagger)^S |{\rm vac}\rangle$$

$$\mathcal{S}_L = -\langle \log \rho \rangle_\rho \rightarrow 2\log(S+1), \quad (L \rightarrow \infty)$$

Boundary Spins: S/2

S	EE	Effective Boundary spins	Degrees of Freedom
1	2 Log 2	$S_{eff}=1/2$	$2^2=4$
2	2 Log 3	$S_{eff}=1$	$3^2=9$
S	2 Log (S+1)	$S_{eff}=S/2$	$(S+1)^2$

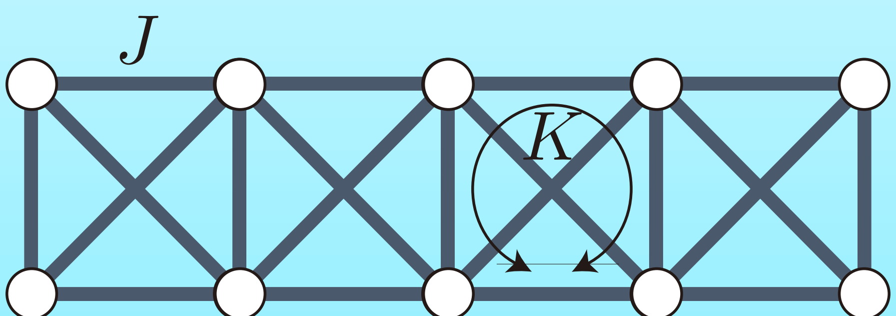
☆ *Fractionalization : Emergent as edge states*

(Quantum Resources for qbits)

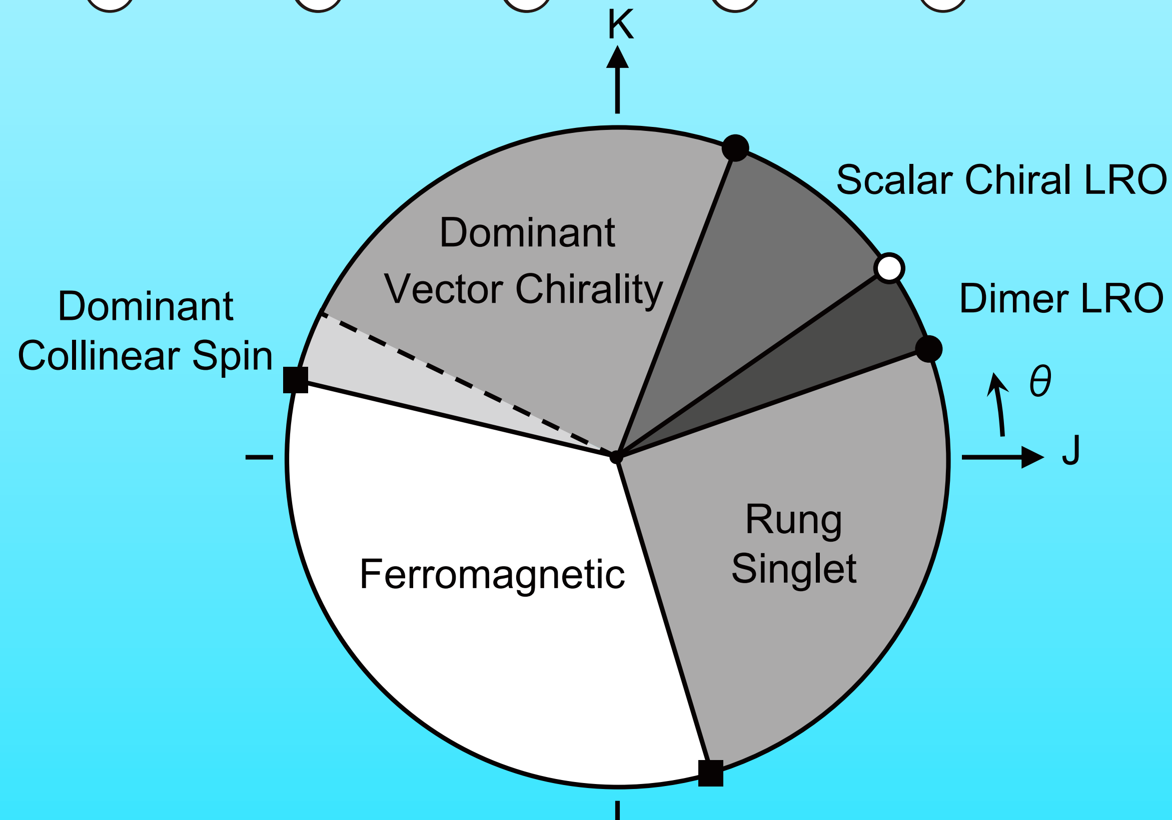
Another Models

Spin ladder model with four-spin cyclic exchange

$$\mathcal{H} = \sum_i \{ J_{\rm r} \boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{2,i} + J_{\rm l} (\boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{1,i+1} + \boldsymbol{S}_{2,i} \cdot \boldsymbol{S}_{2,i+1}) + K (P_i + P_i^{-1}) \}$$



$$\begin{aligned}(P_i + P_i^{-1}) &= \boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{2,i} + \boldsymbol{S}_{1,i+1} \cdot \boldsymbol{S}_{2,i+1} + \boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{1,i+1} \\ &+ \boldsymbol{S}_{2,i} \cdot \boldsymbol{S}_{2,i+1} + \boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{2,i+1} + \boldsymbol{S}_{2,i} \cdot \boldsymbol{S}_{1,i+1} \\ &+ 4(\boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{2,i})(\boldsymbol{S}_{1,i+1} \cdot \boldsymbol{S}_{2,i+1}) \\ &+ 4(\boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{1,i+1})(\boldsymbol{S}_{2,i} \cdot \boldsymbol{S}_{2,i+1}) \\ &- 4(\boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{2,i+1})(\boldsymbol{S}_{2,i} \cdot \boldsymbol{S}_{1,i+1}).\end{aligned}$$



A. Lauchli, G. Schmid and M. Troyer (2003)

We set parameters as

$$\begin{cases} J = J_r = J_l = \cos\theta \\ K = \sin\theta \end{cases}$$

Self dual at the point of $J = 2K$

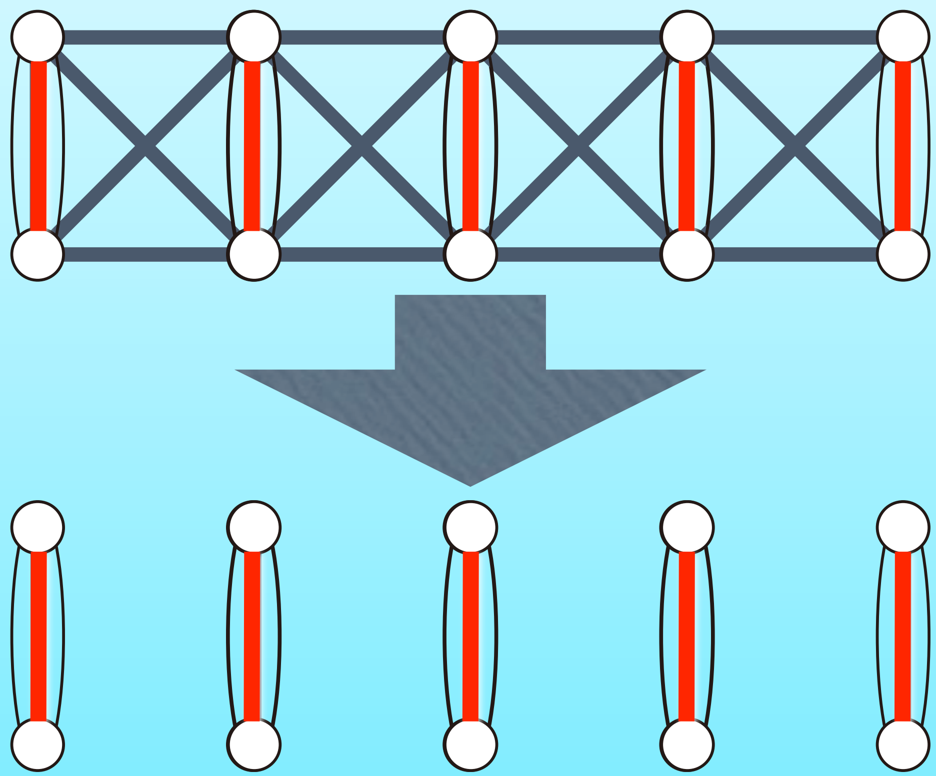
T. Hikihara, T. Momoi and X. Hu (2003)

Adiabatic deformation

I. Maruyama, T. Hirano, YH, arXiv:0806.4416

Rung singlet phase

$\theta = 6$

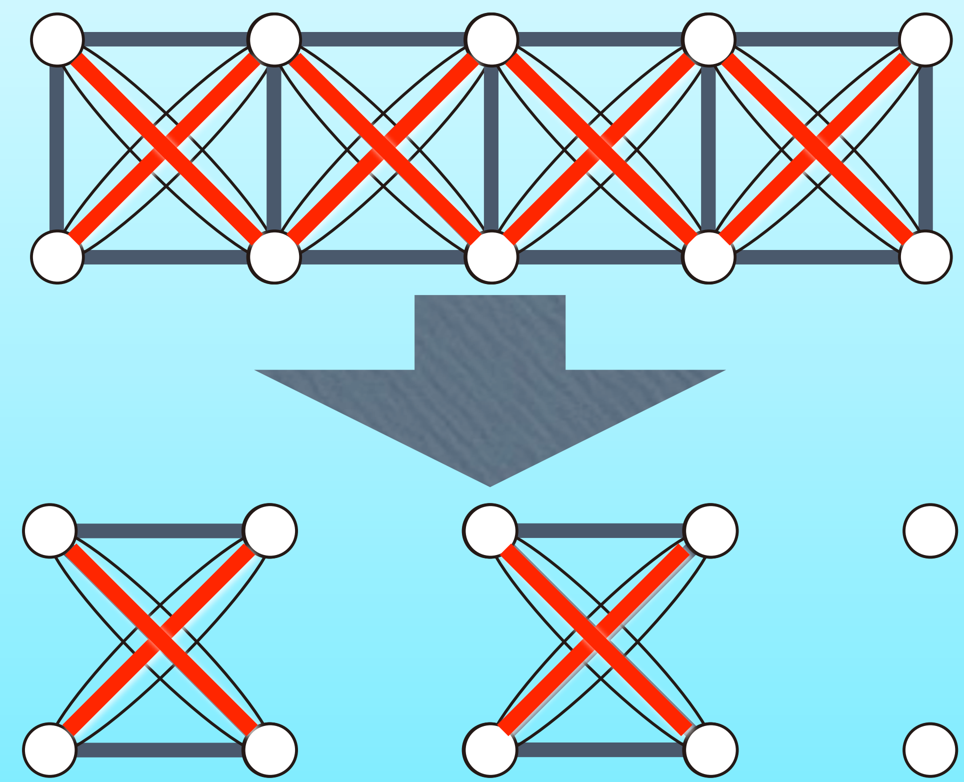


$$H_r = \sum_{i=1} \boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{2,i}$$

Rung singlets

Vector chirality phase

$\theta = 2.6$



$$H_{ps} = \sum_{i \in odd} (\boldsymbol{S}_{1,i} \times \boldsymbol{S}_{2,i}) \cdot (\boldsymbol{S}_{1,i+1} \times \boldsymbol{S}_{2,i+1})$$

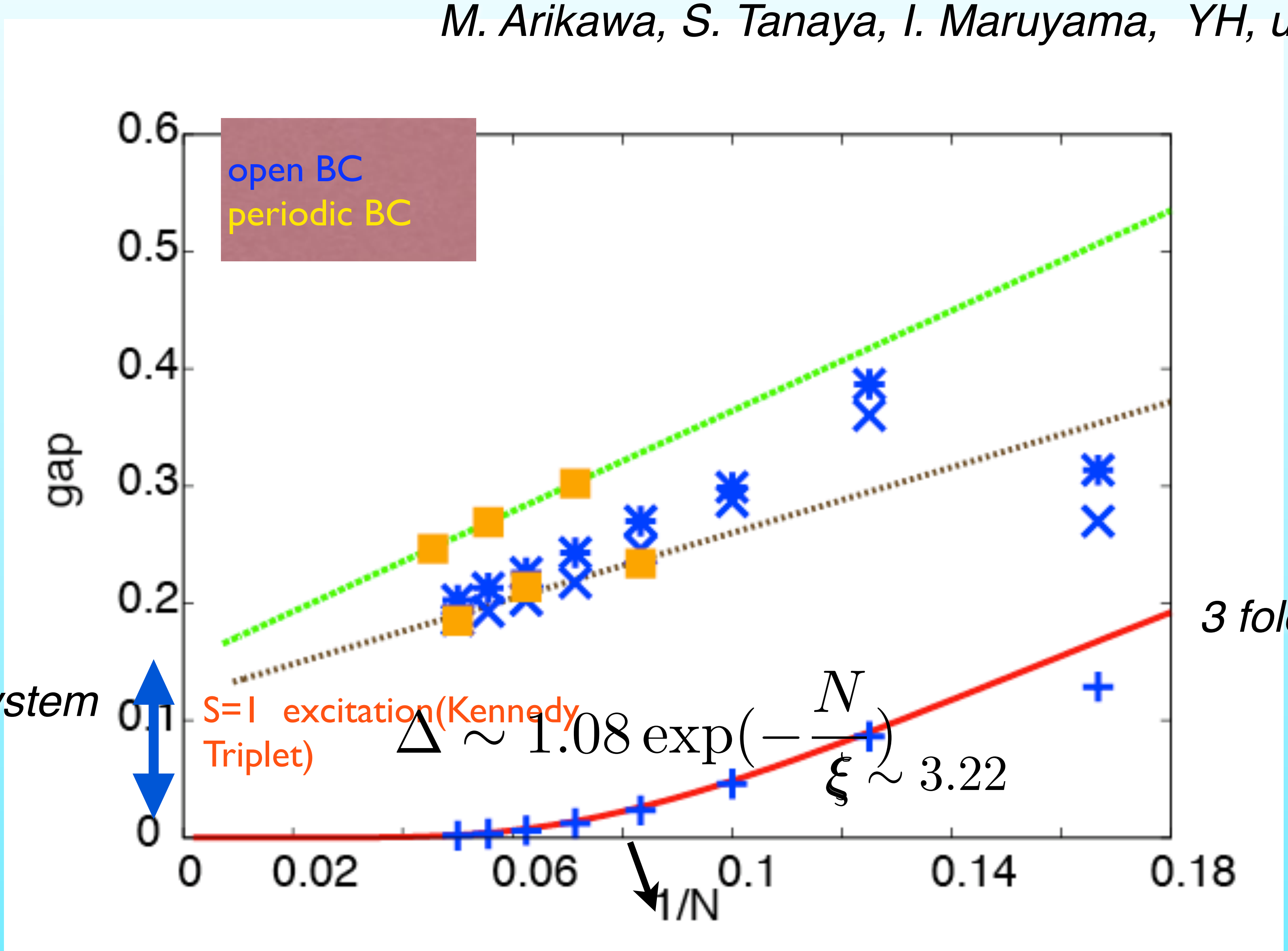
Plaquette singlet (PS)

Berry phase remains the same

Topologically equivalence

Energy spectrum with boundaries (diagonal)

M. Arikawa, S. Tanaya, I. Maruyama, YH, unpublished



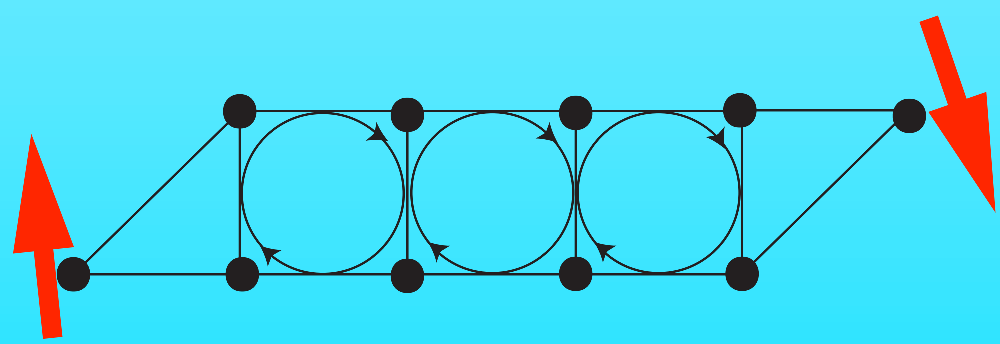
3 fold degenerate

gap for periodic system

Kennedy '90

Ex. for Haldane spin chain

M. Hagiwara, K. Katsumata, I. Affleck, and B. Halperin, '90



Interaction between effective boundary spins

$$H_{eff} = \Delta \boldsymbol{S}_R \cdot \boldsymbol{S}_L$$

Bulk-Edge correspondence for spins

$$\Theta_N^2 = 1 \qquad \Theta_N = (i\sigma_y^1)\otimes(i\sigma_y^2)\cdots(i\sigma_y^N)K$$

K : complex conjugate

Bulk: Z_2 Berry phases

⇅

Edge: Entanglement Entropy

& low energy states in the gap

$S=1/2$ *is always fundamental (electron spin)*

$$\Theta_R{}^2 = -1$$
$$\Theta_L{}^2 = -1$$

$$\Theta_{edges} = \Theta_L \otimes \Theta_R$$

Global TR Θ_N → **Local (edge) TR** $\Theta_L,\ \Theta_R$

Bulk-Edge correspondence for spins

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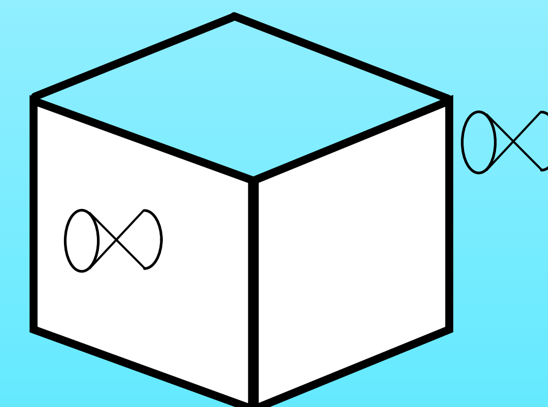
3D ?



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Thank you